1. Determine, with proof, whether each of the following is a vector space. (Note that many of the spaces below are subsets of known vector spaces such as \( \mathbb{R}^n := \{ f : \mathbb{R} \to \mathbb{R} \}, \) so you need only apply the subspace criteria.)

- \( \emptyset \), the empty set.
- \( \{ 0 \} \).
- \( \mathbb{N} := \{ 1, 2, 3, \ldots \} \), the natural numbers.
- \( \mathbb{Z} := \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \), the integers.
- \( \mathbb{Q} := \{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \} \), the rational numbers.
- \( \mathbb{R} \), the real numbers.
- \( \mathbb{C} := \{ a + bi : a, b \in \mathbb{R} \} \), the complex numbers.
- \( M_{m \times n} \), the set of \( m \times n \) matrices.
- \( M_n^* \), the set of invertible \( n \times n \) matrices.
- \( M_{n, \mathbb{Z}} \), the set of \( n \times n \) matrices with integer determinant.
- \( \mathbb{P} := \{ f : \mathbb{R} \to \mathbb{R} : f = a_n x^n + \cdots + a_1 x + a_0 \} \), the set of polynomials.
- \( L^\infty(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} : \exists B : | f(x) | < B \ \forall x \} \), the set of bounded functions.
- \( L^\infty_c(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} : \exists B : f(x) < B \ \forall x \} \), the set of all functions bounded above.

2. Show that the following functions are linear transformations:

- \( E_a : \mathbb{R}^n \to \mathbb{R} \) by \( E_a(f) = f(a) \).
- \( \int_a^b C[a, b] \to \mathbb{R} \) by \( \int_a^b f(x)dx \).

3. A linear transformation from \( V \) to \( \mathbb{R} \) is called a linear functional on \( V \). Prove that \( V^* \), the set of all linear functionals on \( V \), is a vector space. \( V^* \) is called the dual space of \( V \).

4. Let \( U \) and \( V \) be vector spaces, and let \( \text{Hom}_\mathbb{R}(U, V) \) be the set of all linear transformations from \( U \) to \( V \). Show that \( \text{Hom}_\mathbb{R}(U, V) \) is a vector space. Which of the spaces above is isomorphic to \( \text{Hom}_\mathbb{R}(\mathbb{R}^m, \mathbb{R}^n) ? \)

5. Let \( U \) and \( V \) be subspaces of the vector space \( W \). Which of the following are also subspaces?

- \( U \cup V \)
- \( U \cap V \)
- \( U + V := \{ u + v : u \in U, v \in V \} \).
6. Let $T : M_{2 \times 2} \to \mathbb{R}$ be defined by
\[ T(A) = \det A. \]
Is $T$ a linear transformation? Why or why not?

7. Let $A$ be an $n \times (n - 1)$ matrix, and define $T : \mathbb{R}^n \to \mathbb{R}$ by
\[ T(x) = \det[A|x]. \]
Is $T$ a linear transformation? Why or why not? (Hint: consider the cofactor expansion along the last column.)

8. Define the set $S$ by
\[ S = \{ x \in \mathbb{C} | e^x = 1 \}. \]
Is $S$ a vector subspace of $\mathbb{C}$? Why or why not?

9. Let $\mathbb{P}^n$ be the real vector space consisting of all polynomials of degree $n$ or lower with real coefficients. Define the map $T : \mathbb{P}^n \to \mathbb{P}^{n-1}$ by $T(p(x)) = p'(x)$.
   (a) Prove that $T$ is a linear transformation.
   (b) Is $T$ onto, why or why not?
   (c) Is $T$ one to one, why or why not?
   (d) Prove that $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{P}^n$.
   (e) Using $B_1 = \{1, x, \ldots, x^n\}$ as a basis for $\mathbb{P}^n$ and $B_2 = \{1, x, \ldots, x^{n-1}\}$ as a basis for $\mathbb{P}^{n-1}$, give the matrix which represents $T$ with respect to $B_1$ and $B_2$.
   (f) Find a basis for $\text{Ker}(T)$. What is dim $\text{Ker}(T)$?
   (g) Find a basis for $\text{Im}(T)$. What is dim $\text{Im}(T)$?
   (h) What is the rank of $T$?

10. For each matrix below, determine its rank and the dimensions of its nullspace and columnspace. Find bases for its nullspace and columnspace. Find its eigenvalues and the corresponding eigenvectors. Determine if it is diagonalizable.

\[
A := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 5 & 5 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix}
\]

\[
D := \begin{pmatrix} 3 & 0 & 0 \\ 3 & 8 & 0 \\ 4 & 7 & 7 \end{pmatrix}, \quad E := \begin{pmatrix} 2 & 3 & 3 & 1 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad F := \begin{pmatrix} 8 & 3 & 3 & 5 \\ 1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]