§1.1, #5
Consider the matrix
\[
\begin{bmatrix}
1 & -4 & 5 & 0 & 7 \\
0 & 1 & -3 & 0 & 6 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & -5 \\
\end{bmatrix}
\]
as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

**Solution:** First, create a zero in row 2, column 3 by adding 3 * row 3 to row 2. Second, create a zero in row 1, column 3 by subtracting 5 * row 3 from row 1.

§1.1, #11
Solve the system.
\[
\begin{align*}
x_2 + 4x_3 &= -5 \\
x_1 + 3x_2 + 5x_3 &= -2 \\
3x_1 + 7x_2 + 7x_3 &= 6
\end{align*}
\]

**Solution:** The augmented matrix is
\[
\begin{bmatrix}
0 & 1 & 4 & -5 \\
1 & 3 & 5 & -2 \\
3 & 7 & 7 & 6 \\
1 & 3 & 5 & -2 \\
\end{bmatrix}
\]
Swapping rows 1 and 2 yields
\[
\begin{bmatrix}
1 & 3 & 5 & -2 \\
0 & 1 & 4 & -5 \\
3 & 7 & 7 & 6 \\
1 & 3 & 5 & -2 \\
\end{bmatrix}
\]
Subtracting 3 * row 1 from row 3 yields
\[
\begin{bmatrix}
1 & 3 & 5 & -2 \\
0 & -2 & -8 & 12 \\
0 & 1 & 4 & -5 \\
0 & 0 & 0 & 2 \\
\end{bmatrix}
\]
Adding 2 * row 2 to row 3 yields
\[
\begin{bmatrix}
1 & 3 & 5 & -2 \\
0 & 1 & 4 & -5 \\
0 & 0 & 0 & 2 \\
\end{bmatrix}
\]
The bottom row corresponds to the equation 0=2, which has no solutions. Thus the system is inconsistent; it has no solution.
§1.1, #20
Determine the value(s) of $h$ such that the matrix $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$ is the augmented matrix of a consistent linear system.

**Solution:** Row reduction yields the echelon matrix $\begin{bmatrix} 1 & h & -3 \\ 0 & 4 + 2h & 0 \end{bmatrix}$. The system is consistent so long as this matrix has no pivot in the last column. But the 3 in the (first row, third column) cannot be a pivot because nonzero entries occur to its left, and the 0 in the (second row, third column) can never be a pivot. Thus we don’t need any conditions on $h$ to ensure that the system is consistent; the system is consistent for all values of $h$.

§1.1, #24
True or false.
(a) Elementary row operations on an augmented matrix never change the solution set of the associated linear system.
   This is **true**; it follows immediately from the definition and discussion of elementary row operations on pages 7 and 8.

(b) Two matrices are row equivalent if they have the same number of rows.
   This is **false**; consider for example the matrices $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \end{bmatrix}$. They have the same number of rows, but if they were row equivalent then the corresponding systems $x_1 = 1$ and $x_2 = 0$ would have the same solution set (see the box at the top of page 8); they clearly do not.

(c) An inconsistent system has more than one solution.
   This is **false**. An inconsistent system, by the definition on page 4, has no solutions.

(d) Two linear systems are equivalent if they have the same solution set.
   This is **true**. It’s exactly the definition of equivalence given on page 3.

§1.1, #27
Suppose the system $\begin{cases} x_1 + 3x_2 = f \\ cx_1 + dx_2 = g \end{cases}$ is consistent for all possible values of $f$ and $g$. What can you say about the coefficients $c$ and $d$?

**Solution:** Row-reducing the augmented matrix yields $\begin{bmatrix} 1 & 3 & f \\ 0 & d - 3c & g - fc \end{bmatrix}$. The system is consistent for all $f$ and $g$, so the $g - fc$ in the (second row, third column) is never a pivot. Thus, for all $f$ and $g$, either $d - 3c \neq 0$ or $g - fc = 0$. But, regardless of $c$ and $d$, we can always find an $f$ and $g$ so that $g - fc \neq 0$, so it must be the case that $d - 3c \neq 0$. 

2
§1.2, #2c

Determine whether the matrix
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
is in reduced echelon form or only in echelon form.

**Solution:** It is neither. The leading entry of row 2 is in column 1, the same column as the leading entry of row 1. This violates property 2 in the definition on page 14.

§1.2,#4

Row reduce the matrix
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
3 & 5 & 7 & 9 \\
5 & 7 & 9 & 1 \\
\end{bmatrix}
to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

**Solution:** Subtracting 3 * row 1 from row 2 gives
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & −4 & −8 & −12 \\
5 & 7 & 9 & 1 \\
\end{bmatrix}.
\]
Subtracting 5 * row 1 from row 3 gives
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & −4 & −8 & −12 \\
0 & −8 & −16 & −34 \\
\end{bmatrix}.
\]
Dividing row 2 by −4 yields
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & 1 & 2 & 3 \\
0 & −8 & −16 & −34 \\
\end{bmatrix}.
\]
Adding 8 * row 2 to row 3 yields
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & −10 \\
\end{bmatrix}.
\]
Dividing row 3 by −10 leaves
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]
Subtracting 3 * row 3 from row 2 leaves
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & 0 & 0 & 1 \\
0 & 1 & 2 & 0 \\
\end{bmatrix}.
\]
Subtracting 7 * row 3 from row 1 leaves
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & 0 & 0 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]
Finally, subtracting 3 * row 2 from row 1 gives us
\[
\begin{bmatrix}
1 & 0 & −1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]
which is in reduced echelon form. The pivot columns are **columns 1, 2, and 4**; the pivot positions are (row 1, column 1), (row 2, column 2), and (row 3, column 4).
Suppose the matrix \[
\begin{bmatrix}
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast \\
0 & 0 & \ast & 0 \\
\end{array}
\end{bmatrix}
\]
represents the augmented matrix for a system of linear equations. Determine if the system is consistent. If the system is consistent, determine if the solution is unique.

**Solution:** There is no pivot in the last column, so the system is consistent. There are pivots in every other column, so there are no free variables, so the solution is unique.

Choose \( h \) and \( k \) such that the system \[ \begin{align*}
x_1 + hx_2 &= 2 \\
4x_1 + 8x_2 &= k
\end{align*} \] has (a) no solution, (b) a unique solution, and (c) many solutions.

**Solution:** Row-reducing the augmented matrix yields \[
\begin{bmatrix}
1 & h & 2 \\
0 & 8 - 4h & k - 8
\end{bmatrix}.
\]

(a) There is no solution when there is a pivot in the third column, i.e., when \( 8 - 4h = 0 \) and \( k - 8 \neq 0 \), i.e., when \( h = 2 \) and \( k \neq 8 \).

(b) There is a unique solution when there are pivots in columns one and two, but not in column three, i.e., when \( 8 - 4h \neq 0 \), i.e., when \( h \neq 2 \).

(c) There are many solutions when neither column two nor column three contain a pivot, i.e. when \( 8 - 4h = k - 8 = 0 \), or when \( h = 2 \) and \( k = 8 \).

A system of linear equations with fewer equations than unknowns is sometimes called an *underdetermined system*. Suppose that such a system happens to be consistent. Explain why there must be an infinite number of solutions.

**Solution:** Let \( A \) represent the augmented matrix for the system. We have:

\[
\begin{align*}
The \text{number of pivots of } A & \leq \text{The number of rows of } A \\
& = \text{The number of equations in the system} \\
& \leq \text{The number of unknowns in the system} \\
& = (\text{The number of columns of } A) - 1
\end{align*}
\]

Thus \( A \) has at least two columns without pivots; at least one of these must correspond to a free variable in our system. Since the system is consistent and has a free variable, it must have infinitely many solutions.