1. Degree

Let $S^1 := \{e^{2\pi i \theta} \mid \theta \in \mathbb{R}\} \subset \mathbb{C}$. We define the degree of a continuous map $S^1 \to S^1$ as follows. Let $x_0 \in S^1$ and let $\alpha$ be a path from $1 \in S^1$ to $x_0$.

1. Show that if $\gamma$ is a generator of $\pi_1(S^1, 1)$, then $\hat{\gamma}(\gamma)$ is a generator of $\pi_1(S^1, x_0)$. See Appendix *1.

**Solution:** Let $[f] \in \pi_1(S^1, x_0)$. Then $\hat{\gamma}^{-1}([f]) = [\alpha] * [f] * [\bar{\alpha}] \in \pi_1(S^1, 1)$. Since $\gamma$ is a generator for $\pi_1(S^1, 1)$, there is some $n$ such that $[\alpha] * [f] * [\bar{\alpha}] = \gamma^n$ which implies $[f] = [\bar{\alpha}] * \gamma^n * [\alpha] = (\bar{\alpha} * \gamma * [\alpha])^n = \hat{\gamma}(\gamma)^n$.

2. Show that $\hat{\gamma}(\gamma)$ depends only on $x_0$ but not on paths $\alpha$.

**Solution:** Take another path $\beta$ from $1$ to $x_0$. Then $\beta * \bar{\alpha}$ is a loop at $1$. Now $\hat{\beta}(\gamma) = [\bar{\beta}] * \gamma * [\beta] = [\bar{\beta}] * [\beta * \bar{\alpha}] * \gamma * [\beta * \bar{\alpha}]^{-1} * [\beta] = [\bar{\beta}] * [\beta] * [\bar{\alpha}] * \gamma * [\alpha] * [\bar{\beta}] * [\beta] = \hat{\alpha}(\gamma)$.

By (2), it is OK to write $\gamma_{x_0} := \alpha[\gamma]$ for a path $\alpha$ from $1$ to $x_0$. Now let $h : S^1 \to S^1$ be a continuous map. Let $x_0 \in S^1$ and let $x_1 := h(x_0)$. Define degree of $h$ to be an integer $d$ such that

$$h_*(\gamma_{x_0}) = (\gamma_{x_1})^d.$$ 

It is well-defined because, by (1), $\gamma_{x_1}$ is a generator of $\pi_1(S^1, x_1)$ respectively.

3. Show that $d$ is independent of the choice of $x_0$.

**Solution:** Let $y_0 \in S^1$ and let $\beta$ be a path from $x_0$ to $y_0$. Let $y_1 := h(y_0)$, then $\beta' := h \circ \beta$ is a path from $x_1$ to $y_1$.

$$\gamma_{y_1} = [\bar{\beta}'] * \gamma_{x_1} * [\beta']$$

$$h_*(\gamma_{y_0}) = h_(\gamma_{y_0}) * [\beta'] = [\bar{\beta}'] * h_*(\gamma_{y_0}) * [\beta'] = [\bar{\beta}'] * (\gamma_{x_1})^d * [\beta'] = (\gamma_{x_1})^d.$$ 

4. Show that $d$ is independent of the choice of $\gamma$.

**Solution:** By *1, a generator is either $\gamma$ or $\gamma^{-1}$. We use $\gamma^{-1}$, then

$$(\gamma^{-1})_{x_0} = (\gamma_{x_0})^{-1}, \quad (\gamma^{-1})_{x_1} = (\gamma_{x_1})^{-1}.$$ 

Thus $h_*(\gamma_{y_0})^{-1} = ((\gamma_{x_1})^d)^{-1} = ((\gamma_{x_1})^{-1})^d$.

By (3) and (4), we have defined the degree of a map $h$, which is independent of all choices. Now we consider the properties of this degree:

5. Show that if $h, k : S^1 \to S^1$ are homotopic, they have the same degree.

**Solution:** It follows from Theorem 11.1 Lecture Notes.

6. Show that $\deg h \circ k = \deg h \cdot \deg k$.

**Solution:** It follows from (8) and (7).
(7) Compute the degree of the map $h(z) = z^n$ where $n \in \mathbb{Z}$.

Solution: HW 9 (5).

(8) (Optional) Show that if $h, k : S^1 \to S^1$ have the same degree, then they are homotopic.

Solution:
- Consider $h_\ast : \pi_1(S^1, 1) \to \pi_1(S^1, h(1))$ and $k_\ast : \pi_1(S^1, 1) \to \pi_1(S^1, k(1))$. By the assumption, there is $n$ such that $h_\ast(\gamma) = \gamma^p_{h(1)}$ and $k_\ast(\gamma) = \gamma^p_{k(1)}$.
- Let $\gamma := [p_0]$ where $p : \mathbb{R} \to S^1, t \mapsto e^{2\pi it}$ is the standard map and $1 := [0, 1]$. Consider the following lifting diagram,

$\xymatrix{ I \ar[r]^{\tilde{p}_0} & S^1 \ar[r]^h & S^1 \ar[r]^{p} & \mathbb{R} \ar[d]_p & \tilde{h} \circ \tilde{p}_0 \ar@{|-}[rrrr] & & & k \circ \tilde{p}_0 \ar@{|-}[rrrr] & & & \mathbb{R} \ar[d]_p \\
1 \ar[r] & \tilde{p}_0 \ar[r]_{\tilde{h} \circ \tilde{p}_0} & S^1 \ar[r]_{h} & S^1 \ar[r]_{k} \ar[u]^{\tilde{p}_0} & \mathbb{R} \ar[u]_p & & & \mathbb{R} \ar[u]_p & & & \mathbb{R} \ar[u]_p}
$

$\tilde{h} \circ \tilde{p}_0$ and $k \circ \tilde{p}_0$ are the lifts of $h \circ p|_I$ and $k \circ p|_I$ at $h_0 \in p^{-1}(h(1))$ and $k_0 \in p^{-1}(k(1))$ respectively. Let $h_1 := \tilde{h} \circ \tilde{p}_0(1)$ and $k_1 := k \circ \tilde{p}_0(1)$.
- Since $h_\ast(\gamma) = [h \circ p|_I] = \gamma^p_{h(1)}$ and $k_\ast(\gamma) = [k \circ p|_I] = \gamma^p_{k(1)}$, we have $h_1 - h_0 = k_1 - k_0 = n$.
- I wanted to find a path-homotopy between $h \circ p|_I$ and $k \circ p|_I$, but because the starting points and ending points are different, I can find it. So I will shift one of them. Define $\tilde{h} \circ \tilde{p}_0(s) := \tilde{h} \circ \tilde{p}_0(s) - h_0 + k_0$.

Now it is a path from $k_0$ to $k_1$. You can check it by evaluating at 0 and 1. Thus there is a path-homotopy $F$ from $h \circ p|_I$ to $k \circ p|_I$, because $\mathbb{R}$ is a contractible space.
- Consider $\tilde{h}(x) := h(x) \cdot e^{2\pi it(k_0 - h_0)}$. Then $\tilde{h} \circ \tilde{p}_0$ is the lift of $h \circ p|_I$ at $k_0$. Then we check that $p \circ F$ is a path homotopy from $\tilde{h} \circ \tilde{p}_0$ to $k \circ \tilde{p}_0$. Now since $p \circ F$ is a path-homotopy, it factors through $(p, \text{id})$, giving a homotopy $F$ from $\tilde{h}$ to $k$.
- Now the homotopy from $h$ to $\tilde{h}$ is easy to find:

$G(x, t) := h(x)e^{2\pi it(k_0 - h_0)}$.
- By constructing $F$ and $G$, we have shown that $h$ is homotopic to $k$.

(5) says the degree is a homotopy invariant, i.e. if $h, k$ have the different degrees, they cannot be homotopic to each other. Together with (7) and (8), it classifies all homotopy equivalence classes of maps $S^1 \to S^1$. (6) says associating degrees have a certain algebraic structure.

Appendix

*1 An infinite cyclic group $G$ is a group isomorphic to $(\mathbb{Z}, +)$. We say $g \in G$ is a generator, if $G = \{g^n | n \in \mathbb{Z}\}$. If $g$ is a generator, then $g^{-1}$ is also a generator and any generator is either $g$ or $g^{-1}$.