Write the proofs in complete sentences.

1. Let $p : E \to B$ be a covering map. Let $f, g$ be composable paths in $B$, i.e. $f(1) = g(0)$. If $\tilde{f}, \tilde{g}$ are composable paths lifted from $f, g$, then show that $\tilde{f} \circ \tilde{g}$ is a lifting of $f \circ g$.

2. Show that the fundamental group of a torus $S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ as groups (additive on $\mathbb{Z} \times \mathbb{Z}$). (Hint; generalized the proof of $\pi_1(S^1, b_0) \cong \mathbb{Z}$.)

**Solution**

Let $S^1 := \{e^{i\theta}, \theta \in \mathbb{R}\}$ and $p : \mathbb{R} \to S^1, \theta \mapsto e^{2\pi i \theta}$. Let $p := (p, p) : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$.

Let $[f], [g] \in \pi_1(S^1 \times S^1, (1, 1))$. Let $\tilde{f}, \tilde{g}$ be the lifts of $f$ and $g$ at $(0, 0)$ along $p$ and $f(1) = (n_1, m_1)$ and $g(1) = (n_2, m_2)$. We need to show that if $f \circ g$ is the lift of $f \circ g$ at $(0, 0)$ along $p$, then $\tilde{f} \circ \tilde{g}(1) = (n_1 + n_2, m_1 + m_2)$. Consider $\tilde{g}_1(s) := (n_1, m_1) + \tilde{g}(s)$. Then $\tilde{g}$ is the lift of $g$ at $(n_1, m_1)$ along $p$ by the invariance of $p$ under the shifting ($p(x + (n + m)) = p(x)$).

Since $\tilde{g}_1(0) = (n_1, m_1)$, by (1), $\tilde{f} \circ \tilde{g} = \tilde{f} \circ \tilde{g}_1$. Thus $\tilde{f} \circ \tilde{g}(1) = \tilde{g}_1(1) = (n_1 + n_2, m_1 + m_2).

3. A group $G$ acts on a set $X$ from right if there is an action map $G \times X \to X, (g, x) \mapsto gx$ which satisfies $x = x1_G$ and $x(gh) = (xg)h$. Show that there is a natural right action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$ if $p : E \to B$ is a covering map. (Hint: use $\phi_{e_0}$ in Section 10.2 [L].)

4. Let $B$ be a simply-connected space. Then any covering map $p : E \to B$ with $E$ path-connected, is a homeomorphism.

5. Show that the map $p : S^1 \to S^1, z \mapsto z^n$ induces $p_* : \pi_1(S^1, b) \to \pi_1(S^1, b), [f] \mapsto [f]^n$. In other words, through the isomorphism in Section 10.4 [L], $\mathbb{Z} \to \mathbb{Z}, m \mapsto nm$.

**Solution** Under the isomorphism in Theorem 10.8 [L], it is enough to prove $p_*(1) = n$, since $\mathbb{Z}$ is generated by 1. More concretely, it is enough because $f(m) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = mf(1) = mn$. Let $S^1 := \{e^{2\pi i \theta}\}$ and $\tilde{p} : \mathbb{R} \to S^1, \theta \mapsto e^{2\pi i \theta}$. Then it is clear that $\tilde{p}_1 : I \to S^1$ is a loop at 1 and its lift at 0 $\in \mathbb{R}$ is the inclusion $j : 1 \hookrightarrow \mathbb{R}$.

Thus $\phi_0([\tilde{p}_1]) = j(1) = 1$. $p_*(1) = p_*([\tilde{p}_1]) = [p \circ \tilde{p}_1]$. Then the lift of $p \circ \tilde{p}_1$ at 0 is $p \circ \tilde{p}_1 : I \to \mathbb{R}, s \mapsto ns$ (Check this!). Now $p \circ \tilde{p}_1(1) = n$.

**References**

