Let $C$ and $D$ be compact subspaces of a topological space $X$. Show that $A := C \cup D$ is compact.

**Solution:** Let $\{U_\alpha \cap A\}$ be an open covering of $A$ where $U_\alpha$’s are open sets in $X$. Then $\{U_\alpha \cap C\}$ and $\{U_\alpha \cap D\}$ are open coverings of $C$ and $D$. Since $C, D$ are compact, we find a finite subcoverings $\{V_i \cap C, i = 1, \cdots, n\}$ and $\{W_j \cap D, j = 1, \cdots, m\}$ where $\{V_i\}, \{W_j\} \subset \{U_\alpha\}$. Then $\{V_i \cap A, W_j \cap A, i = 1, \cdots, n, j = 1, \cdots, m\}$ is a finite subcovering of $A$.

(2) Prove the following:

(a) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are quotient maps, then $g \circ f : X \rightarrow Z$ is a quotient map.

(b) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous maps. If $f$ and $g \circ f$ are quotient maps, then $g$ is a quotient map.

**Solution:**

(a) First note that since $f$ and $g$ are surjective, then so is $g \circ f$. Since $g$ and $f$ are quotient maps, we have:

$U$ is open in $Z \iff g^{-1}(U)$ is open in $Y \iff f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in $X$.

Thus $g \circ f : X \rightarrow Z$ is a quotient map.

(b) First note that since $g \circ f$ is surjective, then so is $g$. Since $g$ is continuous we have for any open $U \subset Z$, $g^{-1}(U)$ is open in $Y$. Now given some $V \subset Z$, such that $g^{-1}(V)$ is open in $Y$, we want to show that $V$ must be open in $Z$. Since $f$ is a continuous map, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is open in $X$. Finally, since $g \circ f$ is a quotient map we must have that $V$ is open in $Z$ as needed. Hence, $g$ is a quotient map.

(3) Prove the following theorem: The image of a compact space under a continuous map is compact.

**Solution:** Let $f : X \rightarrow Y$ be a continuous map and $X$ a compact space. Let $\{V_\alpha\}$ be an open covering of $f(X)$. Then $\{f^{-1}(V_\alpha)\}$ is an open covering of $X$ and so we have a finite subcovering $\{f^{-1}(V_\alpha), i = 1, \cdots, n\}$. Then it is easy to see that $\{V_\alpha\}$ covers $f(X)$. Indeed $f(X) = f(\cup_i f^{-1}(V_\alpha)) \subset \cup_i V_\alpha$.

(4) Let $f(x_1, \cdots, x_n)$ be a polynomial in variables $x_1, \cdots, x_n$. Show that the set of solutions to $f = 0$ is a closed set in $\mathbb{R}^n$.

**Solution:** Since $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, the preimage $f^{-1}(0)$ of a closed set $\{0\}$ is closed.

(5) Let $\text{SL}(2, \mathbb{R})$ be the set of all $2 \times 2$ matrices of determinant 1. Define a topology on $\text{SL}(2, \mathbb{R})$ as a subspace of $\mathbb{R}^4$.

(a) Show that it is closed.

(b) Show that it is not compact. Hint: Show that it is not sequentially compact and apply Theorem 28.2 [M]. Or, apply Theorem 27.3 [M].
Solution:
(a) $\text{SL}(2, \mathbb{R})$ is the set of solutions to the polynomial equation $ad - bc = 1$, so it is closed.
(b) Let us identify a $2 \times 2$ matrices
$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$
with a 4-vector $(x, y, z, w)$. Now consider
$$X_n := \begin{pmatrix} 1 \\ 0 \\ 0 \\ n \end{pmatrix}$$
which has determinant 1. Since the corresponding sequence $\{(1, n, 0, 1)\}$ in $\mathbb{R}^4$ does not have any convergent subsequence, $\text{SL}(2, n)$ is not sequentially compact and so by Theorem 28.2 [M], it is not compact. Or the distance between $(1, 0, 0, 1)$ and $(1, n, 0, 1)$ can be arbitrary large, so $\text{SL}_2(\mathbb{R})$ is not bounded. Thus by Theorem 27.3 [M], it is not compact.

(6) Let $\mathbb{C} := \{z = x + iy \mid x, y \in \mathbb{R}\}$ be the set of all complex numbers. By identifying $\mathbb{C} \cong \mathbb{R}^2$, we have a topology on $\mathbb{C}$ induced from the standard topology of $\mathbb{R}^2$. Consider
$$X := \mathbb{C}^2 - \{(0, 0)\} = \{(z, w) \in \mathbb{C}^2 \mid (z, w) \neq (0, 0)\}.$$ 
Define an equivalence relation on $X$ by $(z_1, w_1) \sim (z_2, w_2)$ if $(z_1, w_1) = (\lambda z_2, \lambda w_2)$ for some $\lambda \in \mathbb{C} - \{0\}$. The quotient space $X/ \sim$ is called the projective space $\mathbb{CP}^1$. Show that $\mathbb{CP}^1$ is compact. Hint: find a surjective continuous map $S^3 \to \mathbb{CP}^1$ and use the fact that $S^3$ is compact.

Solution: There is $S^3$ sitting in $\mathbb{C}^2 - \{(0, 0)\}$, defined by $|z|^2 + |w|^2 = 1$. We have the following commutative diagram of continuous maps:

$$\begin{array}{ccc}
S^3 & \xrightarrow{c} & \mathbb{C}^2 - \{(0, 0)\} \\
\downarrow p & & \downarrow \pi \\
\mathbb{CP}^1. & & \\
\end{array}$$

$p$ is surjective: let $[z, w] \in \mathbb{CP}^1$. If $|z|^2 + |w|^2 = 1$, then $|rz|^2 + |rw|^2 = r^2$ so that $(rz, rw) \in S^3$. Since $[z, w] = [rz, rw]$, $p$ is surjective. $S^3$ is compact by Theorem 27.3 [M]. $\mathbb{CP}^1$ is its image under a continuous map, so by Theorem 26.5 [M], it is compact.

References