(1) (Set theory) Let \( \mathbb{R} \) be the set of real numbers. Consider the map \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) := x^2 \). Find a maximal subset \( M \) of \( \mathbb{R} \) such that the restriction map \( f|_M \) is injective, i.e. find a subset \( M \) of \( \mathbb{R} \) such that \( f|_M \) is injective and there is no subset of \( \mathbb{R} \) containing \( M \) properly.

**Solution:** \( \mathbb{R}_{\geq 0} = \{ x \in \mathbb{R} \mid x \geq 0 \} \) or \( \mathbb{R}_{\leq 0} = \{ x \in \mathbb{R} \mid x \leq 0 \} \).

(2) (Set theory) Let \( g : X \to Y \) be a map. Let \( A \subset X \) and \( B \subset Y \) be subsets. If \( g \) is injective, then we can show that \( A = g^{-1}(g(A)) \). The proof goes as follows:

- To prove the equality, we need to show that (a) \( A \subset g^{-1}(g(A)) \) and (b) \( g^{-1}(g(A)) \subset A \).
  
  (a) Let \( a \in A \). We need to show that \( a \in g^{-1}(g(A)) \). By the definition of pre-image, we have
  
  \[
  a \in g^{-1}(g(A)) \quad \text{if and only if} \quad g(a) \in g(A).
  \]
  
  The RHS is obviously true and hence the LHS, \( a \in g^{-1}(g(A)) \), is true.
  
  (b) Let \( c \in g^{-1}(g(A)) \subset X \) and then we need to show \( c \in A \). Again by def of pre-image,
  
  \[
  c \in g^{-1}(g(A)) \quad \text{if and only if} \quad g(c) \in g(A).
  \]
  
  RHS means that there is \( a \in A \) such that \( g(a) = g(c) \). However, by the injectivity of \( g \), we have \( c = a \).
  
  Thus \( c \in A \).

Now prove that, if \( g \) is surjective, then \( g(g^{-1}(B)) = B \).

**Solution:** Let \( b \in B \), then clearly by subjectivity \( b \in g(g^{-1}(B)) \), so \( B \subset g(g^{-1}(B)) \). Now let \( c \in g(g^{-1}(B)) \), then \( c = g(a) \) for some \( a \in g^{-1}(B) \) and so \( c = g(a) \in B \), which gives the other inclusion.

(3) (Set theory) Let \( f : X \to Y \) be a map and let \( A_1, A_2 \subset X \) be subsets.

(3.1) Prove the following

(a) \( f(A_1 \cup A_2) \supset f(A_1) \cup f(A_2) \).

(b) \( f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2) \); the equality holds if \( f \) is injective.

(3.1) Define a map \( f : \mathbb{Z} \to \{0, 1\} \) by sending even integers to 0 and odd integers to 1. Find subsets \( A_1 \) and \( A_2 \) of \( \mathbb{Z} \) such that \( f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2) \).

**Solution:**

(3.1) (a) Let \( y \in f(A_1) \cup f(A_2) \). If \( y \in f(A_1) \), then \( \exists x \in A_1 \) such that \( f(x) = y \). Since \( x \in A_1 \), we have \( x \in A_1 \cup A_2 \), so \( y \in f(A_1 \cup A_2) \). Same argument works if \( y \in f(A_2) \).

(b) Let \( y \in f(A_1 \cap A_2) \). Then \( \exists x \in A_1 \cap A_2 \) such that \( f(x) = y \). Since \( x \in A_1 \) and \( x \in A_2 \), we get \( y \in f(A_1) \cap f(A_2) \).

Now assume \( f \) is injective and let \( y \in f(A_1) \cap f(A_2) \). Since \( f \) is injective there is a unique pre-image of \( y \), say \( f(x) = y \). Then \( x \in A_1 \) and \( x \in A_2 \) and so \( y \in f(A_1 \cap A_2) \).

(3.2) The key is that \( f \) is not injective so that two points can go to the same point under \( f \). For example, 0 and 2 go to 0 under \( f \). Let \( A_1 = \{0\} \) and \( A_2 = \{2\} \). See that this gives an example such that \( f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2) \).
(4) **(Topology)** A discrete topology on a set $X$ is given by the collection $\mathcal{T}$ of all subsets. Prove that it satisfies all axioms.

**Solution:** Clear by definition of the power set.

(5) **(Topology)** Let $\mathbb{R}$ be the set of real numbers. Define open sets in $\mathbb{R}$ by subsets that are complement of finite subsets or the whole set $\mathbb{R}$. Check that it defines a topology in $\mathbb{R}$ (known as the finite complement topology).

**Solution:**

(T1) $\mathbb{R} = \emptyset \in \mathcal{T}$ and $\emptyset = \mathbb{R} \in \mathcal{T}$.

(T2) Let $U_i \in \mathcal{T}$ and say $U_i^C = V_i$ for all $i$. Note that $V_i$ are finite subsets or the whole set $\mathbb{R}$. Let $U = \bigcup U_i$ and let $V = U^C = (\bigcup U_i)^C = \bigcap (U_i^C) = \bigcap V_i$. If all of the $V_i$ were $\mathbb{R}$, then $V = \mathbb{R}$ and $U \in \mathcal{T}$, otherwise some $V_i$ is finite and so $V$ is finite, giving $U \in \mathcal{T}$.

(T3) Again let $U_i \in \mathcal{T}$ and say $U_i^C = V_i$ for $1 \leq i \leq n$. Again $V_i$ are finite subsets or the whole set $\mathbb{R}$. Let $U = \bigcap_1^n U_i$ and let $V = U^C = (\bigcap_1^n U_i)^C = \bigcup_1^n (U_i^C) = \bigcup_1^n V_i$. If any of the $V_i$ were $\mathbb{R}$, then $V = \mathbb{R}$ and $U \in \mathcal{T}$, otherwise $V$ is finite since we are taking finite union of finite sets and hence $U \in \mathcal{T}$.

(6) **(Topology)** Show that $\mathbb{R} - \{1/n | n = 1, 2, \cdots\}$ is not an open set in the standard topology of $\mathbb{R}$. You are only allowed to use the materials from Section 1 of the lecture notes.

**Solution:** Let $A := \mathbb{R} - \{1/n | n = 1, 2, \cdots\}$. Notice that $0 \in A$. To show $A$ is open, we need to prove the condition (G1) in Lemma 1.5. [L]. The standard topology is given by $\mathcal{B} = \{ \text{ all open intervals} \}$ (Example 1.7 [L]). Take any open interval containing $0$, say $(a, b)$ where $a < 0 < b$. We can find $n$ such that $0 < 1/n < b$, so that $1/n \in (a, b)$. Thus any interval $(a, b)$ is not a subset of $A$.

(7) **(Topology)** Let $(X, \mathcal{T})$ be a topological space. Prove that a collection $\mathcal{B}$ of open sets is a basis for $\mathcal{T}$ if and only if for every $U \in \mathcal{T}$ and $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

**Solution:** One direction is the content of Lemma 13.2 of [M]. Please see the proof written there. The other direction is the straightforward consequence of the condition (G1) in the Lemma 1.5 of [L].

**References**

