1 Preliminaries

Throughout this section we assume that $B_t, B^1_t, B^2_t, \ldots$ are independent Brownian motions with respect to the same filtration $\mathcal{F}_t$. If $H_t$ is another process we say that $H_t$ is:

- **bounded** if there is an $N < \infty$ such that with probability one, $|H_t| \leq N$ for all $t$;
- **adapted** if each $H_t$ is $\mathcal{F}_t$-measurable;
- **continuous** if with probability one, the function $t \mapsto H_t$ is continuous;
- a **martingale** (with respect to $\mathcal{F}_t$) if for each $t$ $\mathbb{E}[|H_t|] < \infty$ and $\mathbb{E}[H_t | \mathcal{F}_s] = H_s$ for all $s \leq t$;
- a **square integrable martingale** if $H_t$ is a martingale with $\mathbb{E}[H^2_t] < \infty$ for each $t$;
- a **local martingale** if there exists a sequence of stopping times $\tau_1 < \tau_2 < \cdots$ with respect to $\mathcal{F}_t$ such that $\tau_j \to \infty$ and such that for each $j$, $H_{t \wedge \tau_j}$ is a martingale.

Let $\mathcal{I}$ denote the set of adapted, continuous processes. Let $\mathcal{M}, \mathcal{M}^2, \mathcal{LM}$ denote the collection of continuous martingales, square integrable martingales, and local martingales, respectively.

We write $b\mathcal{I}, b\mathcal{M}$ for the collection of bounded processes in $\mathcal{I}, \mathcal{M}$, respectively.

Throughout this section, we will use partitions of intervals $[0, t]$. If $t$ is fixed, we will use $\Pi_n$ to denote a sequence of partitions, i.e., times

$$0 = t_0^n < t_1^n < t_2^n < \cdots < t_k^n \leq t.$$ 

In order to simplify the notation, we will write just $t_j$ for $t^n_j$. We write $||\Pi_n||$ for the mesh of the partition, i.e., the maximal value of $t_j - t_{j-1}$. We write $\overline{\Pi_n}$ for the sequence of dyadic partitions, $t_j = j2^{-n}$ with an appropriate correction for $t_k^n$ if $t$ is not a dyadic rational. Note that $||\overline{\Pi_n}|| = 2^{-n}$.

2 Integration with respect to Brownian motion

We call $H$ a **simple process** if it is of the form

$$H_s = \sum_{j=1}^n C_j 1_{[t_{j-1}, t_j)}(s),$$

where $t_0 < t_1 < \cdots < t_n$ and $C_j$ is a bounded $\mathcal{F}_{t_{j-1}}$-measurable random variable. If $n = 1$, we define

$$Z_s = \int_0^t H_s \, dB_s = \begin{cases} 
 0, & t \leq t_0 \\
 C_1 [B_t - B_{t_0}], & t_0 \leq t \leq t_1 \\
 C_1 [B_{t_1} - B_{t_0}], & t \geq t_1.
\end{cases}$$
For $n > 1$, we define

$$Z_t = \int_t^1 H_s \, dB_s$$

by linearity. It is easy to check that this definition does not depend on how a simple process is written. Let $\mathcal{I}_s$ denote the collection of simple processes.

**Proposition 1.**

- If $H, K \in \mathcal{I}_s$ and $a, b \in \mathbb{R}$, then $aH + bK$ is in $\mathcal{I}_s$ and

$$\int_0^t (aH_s + bK_s) \, dB_s = a \int_0^t H_s \, dB_s + b \int_0^t K_s \, dB_s.$$

- If $H_s \in \mathcal{I}_s$, then

$$Z_t = \int_0^t H_s \, dB_s$$

then $Z_t \in \mathcal{M}^2$. If we define the quadratic variation $\langle Z \rangle_t$ by

$$\langle Z \rangle_t = \int_0^t H_s^2 \, ds,$$

then $Z_t^2 - \langle Z \rangle_t \in \mathcal{M}$. In particular,

$$E[Z_t^2] = E[\langle Z \rangle_t] = \int_0^t E[H_s^2] \, ds. \quad (1)$$

**Proof.** This can be proved directly from the definition and is left to the reader.

The next proposition shows that “quadratic variation” is a good term for $\langle Z \rangle_t$.

**Proposition 2.** If $H \in \mathcal{I}$ and $Z_t = \int_0^t H_s \, dB_s$, and $\Pi_n$ is a sequence of partitions with $\|\Pi_n\| \to 0$, then

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} (Z_{t_j} - Z_{t_{j-1}})^2 = \langle Z \rangle_t$$

in $L^2$ (and hence in probability). If $\Pi_n = \overline{\Pi}_n$ is the sequence of dyadic partitions or any other sequence such that $\sum \|\Pi_n\| < \infty$, then the limit can be taken almost surely.
Proof. Let

$$Q_n = \left[ \sum_{j=1}^{k_n} (B_{t_j} - B_{t_{j-1}})^2 \right] - t = \sum_{j=1}^{k_n} \Delta(j, n)$$

where $\Delta(j, n) = (B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})$. The random variables $\Delta(1, n), \ldots, \Delta(k_n, n)$ are independent and $\Delta(j, n)$ has the distribution of $(t_j - t_{j-1})(N^2 - 1)$ where $N$ is a standard normal. Therefore,

$$\mathbb{E}[Q_n^2] = \text{Var}[Q_n] = \sum_{j=1}^{k_n} (t_j - t_{j-1})^2 \mathbb{E}[(N^2 - 1)^2] \leq \|\Pi_n\| \cdot t \mathbb{E}[(N^2 - 1)^2],$$

and hence $Q_n \to 0$ in $L^2$. Also, Chebyshev’s inequality gives

$$P\{Q_n \geq \epsilon\} \leq \epsilon^{-2} \mathbb{E}[Q_n^2] \leq \epsilon^{-2} \|\Pi_n\| \cdot t \mathbb{E}[(N^2 - 1)^2].$$

If $\|\Pi_n\| < \infty$, the Borel-Cantelli lemma implies that with probability one, for all large $n$, $Q_n \leq \epsilon$. Hence $Q_n \to t$ a.s., for all rational $t$ and hence for all $t$. The gives the result for $H = 1_{[0, t]}$, and the result for other $H \in \mathcal{I}$ follows easily from the definition of the integral.

We will define $\int_0^t H_s \, dB_s$ for $H \in \mathcal{I}$. It will turn out that for fixed $t$,

$$Z_t := \int_0^t H_s \, dB_s = \lim_{n \to \infty} \sum_{j=1}^{k_n} H_{t_{j-1}} [B_{t_j} - B_{t_{j-1}}], \quad (2)$$

where the limit is in probability. This only defines the process up to an event of measure zero. However, we will show that if we restrict to $t$ that are dyadic rationals, $Z_t$ is uniformly continuous on compact intervals with probability one. Hence we can define $Z_t$ for other $t$ by continuity and $(2)$ will still hold.

We first will consider $H \in b\mathcal{I}$. Fix $t$, let $\|H\|_\infty = \sup_{0 \leq s \leq t} \|H_s\|_\infty$, and

$$\text{osc}(H, t, \delta) = \sup \{|H_s - H_r| : 0 \leq r, s \leq t, |r - s| \leq \delta\}.$$ 

Fix $t$. If $H \in b\mathcal{I}$ and $\Pi_n$ is a sequence of partitions with $\|\Pi_n\| \to 0$, let $H_s^{(n)} \in \mathcal{I}$ be defined by $H_s^{(n)} = H_{t_{j-1}}$, $t_{j-1} \leq s < t_j$, and $H_s^{(n)} = 0$ for $s \geq t$. Note that

$$|H_s - H_s^{(n)}| \leq \text{osc}(H, t, \|\Pi_n\|), \quad 0 \leq s < t.$$

Let $O_n = \text{osc}(H, t, \|\Pi_n\|)$. Since the $H_s$ are continuous, $O_n \to 0$ with probability one. Since $O_n \leq 2 \|H\|_\infty$, we also know that $\mathbb{E}[O_n^2] \to 0$. Let

$$Z_t^{(n)} = \int_0^t H_s^{(n)} \, dB_s.$$
Then by (1),
\[ \mathbb{E}[ (Z_t^{(n)} - Z_t^{(m)})^2 ] \leq t \ (O_n + O_m)^2. \]
In particular, \( \{ Z_t^{(n)} \}, n = 1, 2, \ldots, \) is a Cauchy sequence in \( L^2(\Omega, \mathcal{F}_t, \mathbb{P}) \) and we can define
\[ Z_t = \int_0^t H_s \, dB_s = \lim_{n \to \infty} Z_t^{(n)}, \]
where the limit is taken in the \( L^2 \) sense, and hence is only defined up to event of probability zero. In particular (2) holds where the limit can be taken in \( L^2 \) or in probability. Note that
\[ \mathbb{E}[Z_t^2] = \lim_{n \to \infty} \mathbb{E}[ (Z_t^{(n)})^2 ] = \int_0^t \mathbb{E}[H_s^2] \, ds. \]
As before, we define the quadratic variation
\[ \langle Z \rangle_t = \int_0^t H_s^2 \, ds. \]
Note that
\[ Z_t^2 - \langle Z \rangle_t = \lim_{n \to \infty} (Z_t^{(n)})^2 - \langle Z^{(n)} \rangle_t, \]
where the convergence is in \( L^1 \) \( ((Z_t^{(n)})^2 \to Z_t^2 \) in \( L^1 \) and \( \langle Z^{(n)} \rangle_t \) is uniformly bounded and converges almost surely to \( \langle Z \rangle_t \). In particular, if \( s < t \),
\[ \mathbb{E}[Z_t \mid \mathcal{F}_s] = Z_s, \quad \mathbb{E}[Z_t^2 - \langle Z \rangle_t \mid \mathcal{F}_s] = Z_s^2 - \langle Z \rangle_s. \]
Now, for the moment, restrict our consideration to \( t \) in the dyadic rationals \( D \). Using a diagonalization argument, we can find a subsequence \( \Pi_n \), that we will denote as just \( \Pi_n \), such that with probability one for all \( t \in D \),
\[ Z_t = \int_0^t H_s \, dB_s = \lim_{n \to \infty} \sum_{j=1}^{k_n} H_{t_j-1} \, [B_{t_j} - B_{t_{j-1}}]. \]

**Lemma 3.** Suppose \( H \in \mathcal{I} \) and \( Z_t \) is defined as above. Then with probability one, for each \( t \in D \), \( s \mapsto Z_s \) is a uniformly continuous function on \( D \cap [0, t] \).

**Proof.** If \( M_s, s \in D \) is any square integrable martingale, then the \( L^2 \) maximal lemma gives
\[ \mathbb{P}\{ \sup: [M_s - M_0]: 0 \leq s \leq t, s \in D \} \geq \epsilon \leq \epsilon^{-2} \mathbb{E}[M_t^2]. \]
Applying this to \( Z_s - Z_s^{(n)} \), we get that
\[
P\left\{ \sup: |Z_s - Z_s^{(n)}| : 0 \leq s \leq t, s \in D \right\} \geq \epsilon \leq e^{-2 \mathbb{E}[(Z_t - Z_t^{(n)})^2]} \to 0.
\]

By taking a subsequence if necessary, we can assume that \( \sum \mathbb{E}[(Z_t - Z_t^{(n)})^2] < \infty \) and hence, using Borel-Cantelli, that with probability one \( Z_s, 0 \leq s \leq t \), is a uniform limit of \( Z_s^{(n)}, 0 \leq s \leq t \). Since the uniform limit of continuous functions is continuous, the result is proved.

We now define \( Z_t \) for all \( t \) by continuity. It is an easy exercise to verify that this is the same as (2) (up to an event of probability zero). Moreover, if \( \Pi_n \) is any sequence of partitions, then there is a subsequence (which can depend on \( H \) but does not depend on \( \omega \)) such that with probability one, for all \( s \leq t \),
\[
\int_0^s H_r \, dB_r = \lim_{n \to \infty} \sum_{t_j \leq s} H_{t_j - 1} \, [B_{t_j} - B_{t_{j-1}}].
\]

**Proposition 4.** If \( H \in \mathcal{B}, \ Z_t = \int_0^t H_s \, dB_s \), and \( \Pi_n \) is a sequence of partitions as above, then
\[
\lim_{n \to \infty} \sum_{j=1}^{k_n} [Z_{t_j} - Z_{t_{j-1}}]^2 = \langle Z \rangle_t = \int_0^t H_s^2 \, ds
\]
in probability.

**Proof.** Consider a different sequence of partitions \( \Pi^*_n \), which we write \( 0 = s_0 < s_1 < \ldots < s_l = t \). Let \( Z_s^{(l)} \) be the approximation of \( Z_s \) by simple processes using this sequence. For a fixed \( l \),
\[
\lim_{n \to \infty} \sum_{j=1}^{k_n} [Z_{t_j}^{(l)} - Z_{t_{j-1}}^{(l)}]^2 = \langle Z^{(l)} \rangle_t = \int_0^t (H_s^{(l)})^2 \, ds.
\]

But
\[
\| \sum_{j=1}^{k_n} (Z_{t_j} - Z_{t_{j-1}})^2 - [Z_{t_j}^{(l)} - Z_{t_{j-1}}^{(l)}]^2 \|_2 \leq c_l,
\]
for some \( c_l \to 0 \). We note that by taking a subsequence we can make it an almost sure limit for all \( t \in D \), and hence (by monotonicity) for all \( t \).
Let
\[
\phi_N(x) = \begin{cases} 
-N, & x \leq -N \\
0, & -N \leq x \leq N \\
N, & x \geq N.
\end{cases}
\]

Suppose $H \in \mathcal{I}$. Let
\[
Z_{t,N} = \int_0^t \phi_N(H_s) \, dB_s.
\]

We then define the stochastic integral $Z_t$ by
\[
Z_t = \lim_{N \to \infty} Z_{t,N}.
\]

This limit exists trivially (at least on the event where $H$ is continuous). Also, if $\Pi_n$ is any sequence of partitions,
\[
Z_t = \lim_{n \to \infty} \sum_{j=1}^{k_n} H_{t_{j-1}} \left[ B_{t_j} - B_{t_{j-1}} \right],
\]

where the limit is taken in probability. Moreover, we can find a subsquence (depending on $H$) such that the limit is an almost sure limit. As before, let
\[
\langle Z \rangle_t = \int_0^t H_s^2 \, ds.
\]

Then,
\[
\lim_{n \to \infty} \sum_{j=1}^{k_n} [Z_{t_j} - Z_{t_{j-1}}]^2 = \langle Z \rangle_t
\]

in probability. By taking a subsequence, the limit can be made almost sure for every $t$. For later reference, we note that it also follows that if $G_s$ is any continuous process (not necessary adapted), and $s_j \in [t_{j-1}, t_j]$, then
\[
\lim_{n \to \infty} \sum_{j=1}^{k_n} G_{s_j} [Z_{t_j} - Z_{t_{j-1}}]^2 = \int_0^t G_s \, d\langle Z \rangle_s = \int_0^t G_s \, H_s^2 \, ds,
\]

where the limit is in probability (or almost sure along a subsequence).
Proposition 5. If \( H \in \mathcal{I} \), then \( Z_t \) and \( Z_t^2 - \langle Z \rangle_t \) are local martingales. If

\[
E[\langle Z \rangle_t] = \int_0^t E[H_s^2] \, ds < \infty,
\]

then \( Z_t \in \mathcal{M}^2 \), \( Z_t - \langle Z \rangle_t \in \mathcal{M} \).

Proof. If

\[
\tau_N = \inf\{ s : |H_s| \geq N \},
\]

then \( Z_{t\wedge \tau_N} = Z_{t\wedge \tau_N,N} \). Also \( \tau_1 < \tau_2 < \cdots \) and \( \tau_N \to \infty \).

Proposition 6. If \( H, K \in \mathcal{I} \), and \( a, b \in \mathbb{R} \), then

\[
\int_0^t (aH_s + bK_s) \, dB_s = a \int_0^t H_s \, dB_s + b \int_0^t K_s \, dB_s.
\]

Proof. Immediate from the definition.

Proposition 7. If \( H \in \mathcal{I} \), \( Z_t = \int_0^t H_s \, dB_s \), then

\[
Z_t^2 = 2 \int_0^t Z_s \, dB_s + \langle Z \rangle_t.
\]

Proof. Let \( \Pi_n \) be a sequence of partitions as above. Then (recalling that \( Z_0 = 0 \),

\[
Z_t^2 = \sum_{j=1}^{k_n} [Z_t^2_{i_j} - Z_t^2_{i_{j-1}}] = 2 \sum_{j=1}^{k_n} Z_{t_{j-1}} [Z_{i_j} - Z_{i_{j-1}}] + \sum_{j=1}^{k_n} [Z_{i_j} - Z_{i_{j-1}}]^2.
\]

As \( n \to \infty \), the right hand side converges in probability to

\[
2 \int_0^t Z_s \, dZ_s + \langle Z \rangle_t.
\]

Suppose \( H_s, K_s \in \mathcal{I} \) and

\[
Z_t = \int_0^t H_s \, dB_s, \quad Y_t = \int_0^t K_s \, dB_s.
\]

Define

\[
\langle Z, Y \rangle_t = \int_0^t H_s K_s \, ds.
\]
Under this definition, $\langle Z \rangle_t = \langle Z, Z \rangle_t$. Note that

$$4\langle Z, Y \rangle_t = \langle Z + Y \rangle_t - \langle Z - Y \rangle_t.$$ 

This implies that $Z_t Y_t - \langle Z, Y \rangle_t$ is a local martingale and if $\Pi_n$ is a sequence of partitions as above,

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} (Z_{t_j} - Z_{t_{j-1}}) (Y_{t_j} - Y_{t_{j-1}}) = \langle Z, Y \rangle_t,$$

in probability. In fact,

$$Z_t Y_t - \langle Z, Y \rangle_t = \int_0^t Z_s \, dY_s + \int_0^t Y_s \, dZ_s.$$

### 3 Itô’s Formula

We say that $h(t, x)$ is an adapted continuous function if:

- With probability one, $h(t, x)$ is a continuous function on $[0, \infty) \times \mathbb{R}$,
- For each rational $x$, $h(t, x) \in \mathcal{I}$.

The assumption of continuity implies that $h(t, x)$ is determined by the values of $h(t, x)$ for rational $x$. Let $\mathcal{A}$ be the set of adapted continuous functions, and let $\mathcal{A}_{1,2}$ be the set of adapted continuous functions $h(t, x)$ such that with probability one, $\dot{h}, h', h''$ exist and are in $\mathcal{A}$ (here we use dots for time derivatives and $', ''$ for $x$ derivatives). A deterministic function that is $C^1$ in $t$ and $C^2$ in $x$ is in $\mathcal{A}_{1,2}$.

**Proposition 8.** Suppose $Z_t = \int_0^t H_s \, dB_s \in \mathcal{M}$ and $h \in \mathcal{A}_{1,2}$. Then,

$$h(t, Z_t) - h(0, Z_0) = \int_0^t h'(s, Z_s) \, H_s \, dB_s + \int_0^t [h(s, Z_s) + \frac{1}{2} h''(s, Z_s) \, H_s^2] \, ds.$$

**Proof.** We may assume that $h(t, x)$ is zero outside a compact interval in $\mathbb{R}$ (For example, we can let $h_N(t, x) = h(t, x) g_N(x)$ where $g$ is a $C^\infty$ function that is 1 on $[-N, N]$, 0 on $[-N - 1, N + 1]$ and $0 \leq g \leq 1$. If we have the result for each $h_N$ we have the general result..) Fix $t$, and let $\Pi_n$ be a sequence of partitions as above. We write the telescoping sum

$$h(t, Z_t) - h(0, Z_0) = \sum_{j=1}^{k_n} [h(t_{j}, Z_{t_j}) - h(t_{j-1}, Z_{t_{j-1}})].$$
By the mean value theorem, we can write
\[
h(t_j, Z_{t_j}) - h(t_{j-1}, Z_{t_{j-1}}) = \\
[h(t_j, Z_{t_j}) - h(t_{j-1}, Z_{t_{j-1}})] + h'(t_{j-1}, Z_{t_{j-1}}) [Z_{t_j} - Z_{t_{j-1}}] + \frac{1}{2} h''(t_{j-1}, Z_{s_j}) [Z_{t_j} - Z_{t_{j-1}}]^2,
\]
for some \( s_j \in [t_{j-1}, t_j] \).

Let
\[
O_n = \sup \{|\hat{h}(s, x) - \hat{h}(r, x)| : x \in \mathbb{R}, 0 \leq r, s \leq t, |s - r| \leq \|\Pi\|_n\},
\]
\[
O_n^1 = \sup \{|Z_s - Z_r| : 0 \leq r, s \leq t, |s - r| \leq \|\Pi\|_n\},
\]
\[
O_n^2 = \sup \{|h''(r, x) - h''(s, y)| : 0 \leq r, s \leq t, |s - r| \leq \|\Pi\|_n, |x - y| \leq O_n^1\}.
\]

Then \( O_n + O_n^1 + O_n^2 \to 0 \) with probability one. Since
\[
h(t, Z_{t_j}) - h(t_{j-1}, Z_{t_j}) = [\hat{h}(t_{j-1}, Z_{t_{j-1}}) + \epsilon_j] (t_j - t_{j-1}),
\]
for some \( |\epsilon_j| \leq O_n \), it follows that with probability one,
\[
\lim_{n \to \infty} \sum_{j=1}^{k_n} [h(t_j, Z_{t_j}) - h(t_{j-1}, Z_{t_{j-1}})] = \int_0^t \hat{h}(s, Z_s) \, ds.
\]

We have already seen that
\[
\lim_{n \to \infty} \sum_{j=1}^{k_n} h'(t_{j-1}, Z_{t_{j-1}}) [Z_{t_j} - Z_{t_{j-1}}] = \int_0^t h'(s, Z_s) \, dZ_s,
\]
in probability. Finally, since \( |h''(t_{j-1}, Z_{s_j}) - h''(t_{j-1}, Z_{t_{j-1}})| \leq O_n^2 \), it follows from (3) that with probability one
\[
\lim_{n \to \infty} \sum_{j=1}^{k_n} h''(t_{j-1}, Z_{s_j}) [Z_{t_j} - Z_{t_{j-1}}]^2 = \int_0^t h''(s, Z_s) \, d(Z)_s = \int_0^t h''(s, Z_s) \, H_s^2 \, ds.
\]
4 Several Brownian motions

Assume that we have a probability space and filtration $\mathcal{F}_t$ on which are defined a $d$-dimensional Brownian motion $\tilde{B}_t = (B_t^1, \ldots, B_t^d)$. We have already shown how to define the stochastic integral

$$Z_t = \int_0^t H_s \, dB_s^i,$$

where $H_t \in \mathcal{I}$. We have the following covariance rule:

$$\langle \int_0^t H_s \, dB_s^i, \int_0^t K_s \, dB_s^i \rangle_t = \int_0^t H_s \, K_s \, s.$$

**Proposition 9.** If $H, K \in \mathcal{I}$, $i \neq l$, and $Z_t = \int_0^t H_s \, dB_s^i, Y_t = \int_0^t K_s \, dB_s^l$ then $Z_t Y_t$ is a local martingale. Also, if $\Pi_n$ is a sequence of partitions,

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} [Z_{t_j} - Z_{t_{j-1}}] [Y_{t_j} - Y_{t_{j-1}}] = 0,$$

in probability.

**Proof.** Do this first for simple processes, then bounded processes, then general processes. The second result uses the fact that

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} [B_{t_j}^i - B_{t_{j-1}}^i] [B_{t_j}^l - B_{t_{j-1}}^l] = 0,$$

in $L^2$. This follows easily from

$$\mathbb{E} [ [B_{t_j}^i - B_{t_{j-1}}^i] [B_{t_j}^l - B_{t_{j-1}}^l] ] = 0,$$

$$\text{var} [ [B_{t_j}^i - B_{t_{j-1}}^i] [B_{t_j}^l - B_{t_{j-1}}^l] ] = (t_j - t_{j-1})^2.$$

More generally, if

$$Z_t^1 = \sum_{j=1}^d \int_0^t H_s^j \, dB_s^j, \quad Z_t^2 = \sum_{j=1}^d \int_0^t K_s^j \, dB_s^j.$$
are two continuous local martingales, and

\[ \langle Z^1, Z^2 \rangle_t = \int_0^t \sum_{j=1}^d H^j_s K^j_s \, ds, \]

then \( Z^1_t Z^2_t - \langle Z^1, Z^2 \rangle_t \) is a local martingale and

\[ \lim_{n \to \infty} \sum_{j=1}^k_n |Z^1_{t_j} - Z^1_{t_{j-1}}| |Z^2_{t_j} - Z^2_{t_{j-1}}| = \langle Z^1, Z^2 \rangle_t, \]

in probability.

5 Integration with respect to local semimartingales

Let \( \mathcal{SM} \) denote the set of all processes \( Z_t \) of the form

\[ Z_t = \int_0^t R_s \, ds + \sum_{j=1}^d \int_0^t H^j_s \, dB^j_s, \]

where \( R_s, H^j_s \in \mathcal{I} \). These processes are called continuous local *semimartingales*. We write this in shorthand by

\[ dZ_t = R_t \, dt + \sum_{j=1}^d H^j_t \, dB^j_t = R_t \, dt + \bar{H}_t \cdot dB_t, \]

where \( \bar{H}_t = (H^1_t, \ldots, H^d_t), \bar{B}_t = (B^1_t, \ldots, B^d_t) \). The quadratic variation of \( Z_t \) is defined to be the quadratic variation of the martingale part,

\[ \langle Z \rangle_t = \langle \sum_{j=1}^d \int_0^t H^j_s \, dB^j_s \rangle_t = \int_0^t \langle \sum_{j=1}^d (H^j_s)^2 \rangle \, ds \]

If

\[ Y_t = \int_0^t S_s \, ds + \sum_{j=1}^d \int_0^t K^j_s \, dB^j_s, \]

in another local semimartingale in \( \mathcal{SM} \), the covariance process is defined by

\[ \langle Z, Y \rangle_t = \langle \sum_{j=1}^d \int_0^t H^j_s \, dB^j_s, \sum_{j=1}^d \int_0^t K^j_s \, dB^j_s \rangle_t = \int_0^t \langle \sum_{j=1}^d H^j_s K^j_s \rangle \, ds. \]
If $\Pi_n$ is a sequence of partitions as above, then
\[ \lim_{n \to \infty} \sum_{j=1}^{k_n} [Z_{t_j} - Z_{t_{j-1}}] [Y_{t_j} - Y_{t_{j-1}}] = \langle Z, Y \rangle_t, \]
in probability. If $J_s \in \mathcal{I}$, we define the integral with respect to $Z$:
\[ \int_0^t J_s \, dZ_s = \int_0^t J_s \, K_s \, ds + \sum_{j=1}^d \int_0^t J_s \, H_s^i \, dB^i_s. \]
Then,
\[ \langle \int_0^t J_s \, dZ_s, \int_0^t F_s \, dZ_s \rangle_t = \sum_{j=1}^d \int_0^t J_s \, F_s \, (H_s^i)^2 \, ds. \]
Also,
\[ \lim_{n \to \infty} \sum_{j=1}^{k_n} J_{t_j - 1} [Z_{t_j} - Z_{t_{j-1}}] = \int_0^t J_s \, dZ_s, \]
in probability and the following product rule holds,
\[ Z_t Y_t = \int_0^t Z_s \, dY_s + \int_0^t Y_s \, dZ_s + \langle Z, Y \rangle_t. \]
This can be written
\[ d(Z_t Y_t) = Z_t \, dY_t + Y_t \, dZ_t + d\langle Z, Y \rangle_t. \quad (4) \]

6 Itô’s formula for semimartingales

Let $m$ be a positive integer and suppose that
\[ K_s^i, \quad i = 1, \ldots, m; \quad H_s^{i,j}, \quad i = 1, \ldots, m; \quad j = 1, \ldots, d, \]
are in $\mathcal{I}$. Write $\tilde{H}_s^i = (H_s^{i,1}, \ldots, H_s^{i,d})$. Let $Z^1, \ldots, Z^m$ be semimartingales of the form
\[ Z_t^i = Z_t^0 + \int_0^t K_s^i \, ds + \int_0^t \tilde{H}_s^i \cdot dB_s = \int_0^t K_s^i \, ds + \sum_{j=1}^d \int_0^t H_s^{i,j} \, dB^j_s, \]
and let \( \tilde{Z}_t = (Z_1^t, \ldots, Z_m^t) \). Let \( A^m = A \times \cdots \times A \) denote the set of adapted continuous functions \( h(t, x^1, \ldots, x^m) \), and let \( A_{1,2}^m \) denote the set of \( h(t, x^1, \ldots, x^m) \) such that for all \( 1 \leq i, l \leq m \) the derivatives

\[
\dot{h}(t, x^1, \ldots, x^m), \quad h_i(t, x^1, \ldots, x^m), \quad h_{il}(t, x^1, \ldots, x^m)
\]

exist and are in \( A^m \). Here we write \( h_i \) for differentiation with respect to \( x^i \) and \( h_{il} \) for the corresponding double partials. We state an extension of Itô’s formula which can be proved in the same way as the previous one.

**Proposition 10.** Suppose \( Z^1, \ldots, Z^m \) are as above and \( h \in A_{1,2}^m \). Then

\[
h(t, \tilde{Z}_t) - h(0, \tilde{Z}_0) = \int_0^t \dot{h}(s, \tilde{Z}_s) \, ds + \sum_{i=1}^m \int_0^t h_i(s, \tilde{Z}_s) \, dZ^i_s + \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \int_0^t h_{il}(z, \tilde{Z}_s) \, d\langle Z^i, Z^l \rangle_s =
\]

\[
\sum_{j=1}^d \int_0^t \left[ \sum_{i=1}^m h_i(s, \tilde{Z}_s) H_{ij}^s \right] \, dB^j_s +
\]

\[
\int_0^t \left[ \dot{h}(z, \tilde{Z}_s) + \sum_{i=1}^m h_i(s, \tilde{Z}_s) K^i_s + \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^d h_{il}(s, \tilde{Z}_s) H_{ij}^s H_{lj}^s \right] \, ds.
\]

One can generalize this proposition. For \( 1 \leq b \leq m + 1 \), let \( A_{1,2}^{m,b} \) be the set of adapted continuous functions \( h(t, z^1, \ldots, z^m) \) such that for all \( 1 \leq i \leq m \) and \( b \leq i, l \leq m \) the derivatives

\[
\dot{h}(t, x^1, \ldots, x^m), \quad h_i(t, x^1, \ldots, x^m), \quad h_{il}(t, x^1, \ldots, x^m)
\]

exist and are in \( A^m \). Under this definition, \( A_{1,2}^m = A_{1,2}^{m,1} \). Then we get the following.

**Proposition 11.** Suppose \( Z^1, \ldots, Z^m \) are as above and \( h \in A_{1,2}^{m,b} \). Suppose \( H_i \equiv 0 \) for \( i < b \). Then

\[
h(t, \tilde{Z}_t) - h(0, \tilde{Z}_0) = \int_0^t \dot{h}(s, \tilde{Z}_s) \, ds + \sum_{i=1}^m \int_0^t h_i(s, \tilde{Z}_s) \, dZ^i_s + \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \int_0^t h_{il}(z, \tilde{Z}_s) \, d\langle Z^i, Z^l \rangle_s =
\]

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\[
\sum_{j=1}^{d} \int_{0}^{t} \left[ \sum_{i=b}^{m} h_{i}(s, \tilde{Z}_{s}) H^{i,j}_{s} \right] dB^{j}_{s} + \\
\int_{0}^{t} \left[ \tilde{h}(z, \tilde{Z}_{s}) + \sum_{i=1}^{m} h_{i}(s, \tilde{Z}_{s}) K^{i}_{s} + \frac{1}{2} \sum_{i=b}^{m} \sum_{l=b}^{m} \sum_{j=1}^{d} h_{i}(s, \tilde{Z}_{s}) H^{i,j}_{s} H^{l,j}_{s} \right] ds.
\]

If the semimartingales are just independent Brownian motions, the form of Itô's formula is easier. If \( h(t, x^{1}, \ldots, x^{d}) \) is a function of on \([0, \infty) \times \mathbb{R}^{d} \), write \( \Delta \) for the Laplacian in the space variables:

\[
\Delta h(t, \tilde{x}) = \sum_{j=1}^{d} h_{j}(t, \tilde{x}).
\]

**Proposition 12.** Suppose \( \tilde{B}_{t} = (B^{1}_{t}, \ldots, B^{d}_{t}) \) is a standard \( d \)-dimensional Brownian motion and \( h(t, x^{1}, \ldots, x^{d}) \) is a function that is \( C^{1} \) in \( t \) and \( C^{2} \) in \( x^{1}, \ldots, x^{d} \). Then,

\[
h(t, \tilde{B}_{t}) - h(0, \tilde{B}_{0}) = \sum_{j=1}^{d} \int_{0}^{t} h_{j}(s, \tilde{B}_{s}) ] dB^{j}_{s} + \int_{0}^{t} \left[ \tilde{h}(s, \tilde{B}_{s}) + \frac{1}{2} \Delta h(s, \tilde{B}_{s}) \right] ds.
\]

### 7 Time changes of martingales

Suppose \( H \in \mathcal{I} \) and \( Z_{t} = \sum_{j=1}^{d} \int_{0}^{t} H^{j}_{s} dB_{s} \) is a continuous local martingale. Suppose that with probability one,

\[
\lim_{t \to \infty} \langle Z \rangle_{t} = \int_{0}^{\infty} \sum_{j=1}^{d} (H^{j}_{s})^{2} ds = \infty,
\]

and define stopping times \( \tau_{r} \) by

\[
\tau_{r} = \inf\{t : \langle Z \rangle_{t} = r\}.
\]

**Proposition 13.** Let \( W_{r} = Z_{\tau_{r}} \). Then \( W_{r} \) is a standard Brownian motion with respect to the filtration \( \mathcal{F}_{\tau_{r}} \).

**Proof.** Obviously \( W_{0} = 0 \) and \( W_{r} \) has continuous paths almost surely. It suffices, therefore, to show that for every \( r_{0} < r \), that the distribution of \( W_{r} - W_{r_{0}} \) conditioned on \( \mathcal{F}_{r_{0}} \) is normal, mean zero, variance \( r - r_{0} \). From the strong Markov property of Brownian motion, it suffices to prove this when \( r_{0} = 0 \).
If \( y \in \mathbb{R} \), let \( Y_t = \exp \{ iyZ_t - y^2 \langle Z \rangle_t / 2 \} \). Itô’s formula shows that this is a local martingale (here we need to apply Itô’s formula to a complex function, but we just apply it to the real and imaginary parts separately). For \( t \leq \tau \), \( Y_t \) is uniformly bounded. Hence we can use the optional sampling theorem to conclude that \( 1 = \mathbb{E}[Y_0] = \mathbb{E}[Y_{\tau}] \). This implies \( \mathbb{E}[e^{iyZ_{\tau}}] = e^{r^2/2} \) for each \( y \in \mathbb{R} \) and hence \( Z_{\tau} \) has a normal distribution with mean zero and variance \( r \).

8 Examples

8.1 Martingales from harmonic functions

Suppose \( f : \mathbb{R}^d \to \mathbb{R} \) is a harmonic function, i.e., \( \Delta f(\bar{x}) = 0 \) for all \( \bar{x} \in \mathbb{R}^d \). Let \( \tilde{B}_t \) be a standard \( d \)-dimensional Brownian motion. Then Itô’s formula shows that

\[
 f(\tilde{B}_t) - f(\tilde{B}_0) = \sum_{j=1}^{d} \int_0^t f_j(\tilde{B}_s) \, dB_s^j.
\]

In particular, \( Y_t = f(\tilde{B}_t) \) is a local martingale. If \( D \) is an open set in \( \mathbb{R} \) and

\[
 \tau = \tau_D = \inf \{ t \geq 0 : \tilde{B}_t \notin D \},
\]

and \( \Delta f(\bar{x}) = 0 \) for \( \bar{x} \in D \), then we can similarly show that \( Y_{t \wedge \tau} \) is a local martingale. If \( D \) is a bounded domain and \( f \) is continuous on \( \overline{D} \) (so that \( f \) is bounded on \( \overline{D} \)), then \( Y_{t \wedge \tau} \) is a bounded martingale.

8.2 Exponential martingale

Suppose that \( Z_t = \sum_{j=1}^{d} \int_0^t H_s \, dB_s \) is a continuous local martingale. Then applying Itô’s formula with \( h(t, x) = e^x \), gives

\[
e^{Z_t} - 1 = \int_0^t e^{Z_s} \, dZ_s + \int_0^t \frac{1}{2} e^{Z_s} \, d\langle Z \rangle_s.
\]

If \( Y_t = Z_t - \langle Z \rangle_t / 2 \), then

\[
e^{Y_t} - 1 = \int_0^t e^{Y_s} \, dZ_s,
\]

i.e., \( M_t = e^{Y_t} = \exp \{ Z_t - \langle Z \rangle_t / 2 \} \) satisfies the exponential differential equation \( dM_t = M_t \, dZ_t \). In particular, \( M_t \) is a local martingale called the exponential martingale derived from \( Z_t \).
8.3 Bessel process

Let \( \tilde{B}_t \) be a standard \( d \)-dimensional Brownian motion \((d > 1)\) with \( B_0 \neq 0 \). Applying Itô's formula to \( f(x^1, \ldots, x^d) \) gives

\[
|B_t| = |B_0| + \sum_{j=1}^{d} \int_0^t \frac{B^j_t}{|B_s|} \, ds + \int_0^t \frac{d - 1}{2} \frac{1}{|B_s|} \, ds.
\]

Since, with probability one, \( B_t \neq 0 \) for all \( t \), there is no problem with the integrals. Note that

\[
\tilde{B}_t := \sum_{j=1}^{d} \int_0^t \frac{B^j_s}{|B_s|} \, ds
\]

is a continuous local martingale with

\[
\langle \tilde{B} \rangle_t = \sum_{j=1}^{d} \int_0^t \left[ \frac{B^j_s}{|B_s|} \right]^2 \, ds = t.
\]

Hence by Proposition 13, \( \tilde{B}_t \) is a Brownian motion, and \( Y_t := |B_t| \) satisfies the stochastic differential equation

\[
dY_t = d\tilde{B}_t + \frac{a}{Y_t} \, dt,
\]

where \( a = (d - 1)/2 \). This equation is often called the Bessel equation and the solution \( Y_t \) is called a Bessel process. One can solve this equation for all \( a \), at least up to the first time \( \tau \) with \( Y_\tau = 0 \).

8.4 Holomorphic functions

A function \( f : \mathbb{C} \to \mathbb{C} \) is called holomorphic or analytic at \( z \) if the complex derivative exists. Equivalently a holomorphic function is a function \( f : \mathbb{R}^2 \to \mathbb{R}^2, f(x^1, x^2) = (u(x^1, x^2), v(x^1, x^2)) \) that satisfies the Cauchy-Riemann equations, \( u_1(\bar{x}) = v_2(\bar{x}), u_2(\bar{x}) = -v_2(\bar{x}) \). The Cauchy-Riemann equations imply that \( u, v \) are harmonic functions. If \( \tilde{B}_t \) is a standard two-dimensional Brownian motion, we can consider it as a complex valued Brownian motion \( B_t = B^1_t + iB^2_t \). Let \( Y_t = f(B_t) \). Then Itô's formula and the Cauchy-Riemann equations give

\[
d[u(\tilde{B}_t)] = u_1(\tilde{B}_t) \, dB^1_t + u_2(\tilde{B}_t) \, dB^2_t,
\]

\[
d[v(\tilde{B}_t)] = -u_2(\tilde{B}_t) \, dB^1_t + u_1(\tilde{B}_t) \, dB^2_t.
\]
Then $u(\tilde{B}_t)$ and $v(\tilde{B}_t)$ are local martingales with

$$
\langle u(\tilde{B}) \rangle_t = \langle v(\tilde{B}) \rangle_t = \int_0^t ([u_1(\tilde{B}_s)]^2 + [u_2(\tilde{B}_s)]^2) \, ds = \int_0^t |f'(\tilde{B}_s)|^2 \, ds.
$$

Note also that $\langle u(\tilde{B}), v(\tilde{B}) \rangle_t = 0$. A proof similar to that in Proposition 13 can be used to show that the $\sigma_r$ is the first time $t$ that $\langle u(\tilde{B}) \rangle_t = \langle v(\tilde{B}) \rangle_t = r$, then $Y_{\sigma_r}$ is a standard complex Brownian motion.

If $D \subset \mathbb{C}$ is a domain, $f : D \to \mathbb{C}$ is holomorphic and one-to-one in $D$ and

$$
\tau = \tau_D = \inf \{t \geq 0 : \tilde{B}_t \notin D\},
$$

then if $\tilde{B}_0 \in D$, $X_r = Y_{\sigma_r \wedge \tau}$ is a Brownian motion starting at $f(\tilde{B}_0)$, stopped upon first leaving $f(D)$.

## 9 Girsanov’s transformation

Suppose $K_s \in \mathcal{A}$ and let $M_t$ be the positive martingale

$$
M_t = \exp \left\{ \int_0^t K_s \, dB_s - \frac{1}{2} \int_0^t K_s^2 \, ds \right\}.
$$

Itô’s formula shows that $M_t$ satisfies the equation $dM_t = M_t \, K_t \, dB_t$. Let $Q$ denote the measure on $(\Omega, \mathcal{F})$ whose Radon-Nikodym derivative with respect to $P$ is $M_t$. If we let $E_Q$ denote expectations with respect to $Q$, then for every $\mathcal{F}_t$ measurable $Y$, $E_Q[Y] = E[Y \, M_t]$. Note that if $s < t$ and $Y$ is $\mathcal{F}_s$-measurable, then

$$
E[Y \, M_t] = E[E[Y \, M_t \mid \mathcal{F}_s]] = E[Y \, E[M_t \mid \mathcal{F}_s]] = E[Y \, M_s],
$$

so the definition is consistent.

**Proposition 14.** If

$$
X_t = B_t - \int_0^t K_r \, dr,
$$

then $X_t$ is a $Q$-martingale with respect to $\mathcal{F}_t$, i.e., if $s < t$, then $E_Q[X_t \mid \mathcal{F}_s] = X_s$.

**Proof.** We first recall that the conditional expectation $E_Q[X_t \mid \mathcal{F}_s]$ is the unique $\mathcal{F}_s$-measurable random variable $Y$ such that for all $A \in \mathcal{F}_s$, $E_Q[Y \, 1_A] = E_Q[X_t \, 1_A]$. In other words,

$$
E[Y \, 1_A \, M_s] = E[X_t \, 1_A \, M_t].
$$

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Hence to prove the result, it suffices to show that $X_t M_t$ is a $\mathbf{P}$-martingale. The product formula gives

$$d(X_t M_t) = X_t \, dM_t + M_t \, dX_t + d\langle X, M \rangle_t.$$  

The “$dt$” terms cancel which establishes the proposition.

More generally, let

$$Z_t = \sum_{j=1}^{d} \int_{0}^{t} H_{s}^{j} \, dB_{s}^{j},$$

and let $M_t = \exp \{Z_t - \langle Z \rangle_t / 2\}$ be the corresponding exponential martingale satisfying $dM_t = M_t \, dZ_t$. Assume sufficient boundedness so that $M_t$ is a martingale (not just a local martingale).

**Proposition 15.** If

$$X_t = Z_t - \int_{0}^{t} M_{r}^{-1} \, d\langle Z \rangle_r,$$

then $X_t$ is a $\mathbf{Q}$-martingale with respect to $\mathcal{F}_t$, i.e., if $s < t$, then $\mathbf{E}_Q[X_t \mid \mathcal{F}_s] = X_s$. 
