MATH1110 Summer 2013 Prelim 1 Solution

July 10, 2013

1

(a) No. It fails the horizontal line test. For example, the line $y = 1$ intersect the graph in 5 points.

(b) $[-2, 0], (1, 2)$

(c) Since $\lim_{x \to 1^-} f(x) = 1$ while $\lim_{x \to 1^-} f(x) = 0.4$. They do not agree and hence the limit does not exist.

(d) They are at $x = 1$ and $x = 2$.

(e) At $x = 2$.

2

(a) $\lim_{x \to 1} \log_2(x^2 + 3) = \log_2(1 + 3) = \log_2(4) = 2$

(b) $\lim_{x \to \frac{\pi}{4}^+} \tan x = \lim_{x \to \frac{\pi}{4}^+} \frac{\sin x}{\cos x} = \frac{1^+}{0^-} = -\infty$

(c) $\lim_{x \to 0} \frac{\sin x + 1}{\cos x} = \frac{0 + 1}{1} = 1$

(d) $\lim_{x \to \infty} \frac{x \sqrt{x^2 + 2x}}{3x^2 - 5x + 1} = \lim_{x \to \infty} \frac{x \sqrt{x^2 + 2x}}{3x^2 - 5x + 1} = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{2}{x}}}{3 - \frac{5}{x} + \frac{1}{x^2}} = \frac{\sqrt{1}}{3} = \frac{1}{3}$

3

(a) True

(b) True
(c) False
(d) True
(e) False

4

(a) Both the functions \( g_1(x) = x \) and \( g_2(x) = \ln x \) are continuous functions. Hence so is their product \( g(x) = g_1(x)g_2(x) = x \ln x \). The domain of \( g_1 \) is the entire real line, while the domain of \( g_2 \) is the half real line \((0, \infty)\), hence the domain of \( g \) is \((0, \infty)\).

(b) The function \( h(x) = e^x \) is continuous. Hence \( h(g(x)) = e^{x \ln x} \) is also continuous in \( x \). Moreover, \( e^{x \ln x} = (e^{\ln x})^x = x^x \).

(c) Note that \( f(1) = 1^1 = 1 \) and \( f(2) = 2^2 = 4 \). For this part we use the Intermediate Value Theorem: a continuous function (and we just proved that \( x^x \) is continuous) that goes from \( f(1) = 1 \) to \( f(2) = 4 \) must sweep through all the values in between, in particular, it must sweep through the value \( y = 3 \). This means there is some \( x, 1 < x < 2 \) such that \( f(x) = 3 \).

5

(a) It takes 1 hour to reach the top. The height at that point is

\[
H(1) = 75 - 60 \cos(\pi) = 75 + 60 = 135 \text{m}
\]

(b) The average rate of change is

\[
\frac{H(\frac{1}{2} + h) - H(\frac{1}{2})}{h} = \frac{(75 - 60 \cos(\frac{2\pi}{3})) - (75 - 60 \cos(\pi/2))}{\frac{1}{6}} = 360(-\cos 2\pi/3) = 360 \sin(\pi/6) = 180
\]

(c) First we simplify

\[
\frac{H(\frac{1}{2} + h) - H(\frac{1}{2})}{h} = \frac{60(\cos(\frac{\pi}{2}) - \cos(\frac{\pi}{2} + \pi h))}{h} = \frac{60 \sin(\pi h)}{h} \times \pi
\]

Hence \( \lim_{h \to 0} \frac{H(\frac{1}{2} + h) - H(\frac{1}{2})}{h} = \lim_{h \to 0} \frac{60 \sin(\pi h)}{\pi h} \times \pi = 60\pi \).
(a)

\[
\begin{align*}
  f(1) &= \lim_{t \to \infty} \frac{1}{1 + t} = \frac{1}{\infty} = 0 \\
  f(2) &= \lim_{t \to \infty} \frac{1}{1 + 4t} = \frac{1}{\infty} = 0 \\
  f(0) &= \lim_{t \to \infty} \frac{1}{1} = 1
\end{align*}
\]

(b) For any \( a \neq 0 \), we have \( a^2 t \to \infty \) as \( t \to \infty \). Hence:

\[
    f(a) = \lim_{t \to \infty} \frac{1}{1 + a^2 t} = \frac{1}{\infty} = 0
\]

(c) By (b) and by calculation of \( f(0) \) at (a), \( y = f(x) \) always stays at \( y = 0 \), except when \( x = 0 \) where it has a jump up to \( y = 1 \). Hence, it has a jump discontinuity at \( x = 0 \).