Solution to Midterm Exam

July 15, 2015

1. Compute the following limits, justifying your steps. L’Hopital’s Rule is NOT allowed.

(a) \( \lim_{x \to \pi} e^{\frac{1}{x} + \cos(x)} \)

\[ \text{Sol:} \quad \text{Since } e^x \text{ is continuous in } \mathbb{R} \text{ and } \frac{1}{x} + \cos(x) \text{ is continuous at } x = \pi, \]
\[ \lim_{x \to \pi} e^{\frac{1}{x} + \cos(x)} = e^{\lim_{x \to \pi} \left( \frac{1}{x} + \cos(x) \right)} = e^{\left( \frac{1}{\pi} - 1 \right)}. \]

(b) \( \lim_{x \to -\infty} \frac{x + 1}{\sqrt{x^2 + 1}} \)

\[ \text{Sol:} \quad \text{Let } y = -x, \text{ so } x = -y, \text{ and } x \to -\infty \text{ is equivalent to } y \to \infty. \]
\[ \lim_{x \to -\infty} \frac{x + 1}{\sqrt{x^2 + 1}} = \lim_{y \to \infty} \frac{-y + 1}{\sqrt{(-y)^2 + 1}} = \lim_{y \to \infty} \frac{-y + 1}{\sqrt{y^2 + 1}} = \lim_{y \to \infty} \frac{-1 + \frac{1}{y}}{\sqrt{1 + \frac{1}{y^2}}} = \frac{-1}{1} = -1. \]

(c) \( \lim_{x \to \pi} \frac{\sin(x) - \sin(\pi)}{\pi - x} \)

\[ \text{Sol:} \quad \text{Let } h = x - \pi, \text{ then } x = \pi + h, \text{ and } x \to \pi \text{ is equivalent to } h \to 0. \]
\[ \lim_{x \to \pi} \frac{\sin(x) - \sin(\pi)}{\pi - x} = \lim_{h \to 0} \frac{\sin(\pi + h) - \sin(\pi)}{-h} \]
\[ = -\lim_{h \to 0} \frac{-\sin(h)}{h} = -\frac{d}{dx}|_{x=\pi} \sin(x) = -\cos(\pi) = 1. \]
2. (20 pts) Compute the following derivatives:

(a) \( \frac{d}{dx} (x^3 e^x) = 3x^2 e^x + x^3 e^x \), using the Product Rule.

(b) \( \frac{d}{d\theta} (\tan(2\theta + 1) - 3\theta + e^2) = 2\sec^2(2\theta + 1) - 3 \)

(c) \( \frac{d}{dy} \left( \frac{\sin(y) + 1}{y + 1} \right) = \frac{\cos(y)(y + 1) - (\sin(y) + 1)}{(y + 1)^2} \), using the Quotient Rule.

(d) \( \frac{d}{dx} (x^{\cos(x)} + \ln(x)) = \frac{d}{dx} (x^{\cos(x)}) + \frac{1}{x} \)

To find \( \frac{d}{dx} (x^{\cos(x)}) \) we use logarithmic differentiation. Let \( y = x^{\cos(x)} \). Then

\[ \ln(y) = \ln(x^{\cos(x)}) = \cos(x) \ln(x) \]

Differentiate both sides:

\[ \frac{1}{y} \frac{dy}{dx} = \frac{\cos(x)}{x} - \sin(x) \ln(x) \]

and therefore

\[ \frac{dy}{dx} = y \left( \frac{\cos(x)}{x} - \sin(x) \ln(x) \right) = x^{\cos(x)} \left( \frac{\cos(x)}{x} - \sin(x) \ln(x) \right) \]

Therefore \( \frac{d}{dx} (x^{\cos(x)} + \ln(x)) = x^{\cos(x)} \left( \frac{\cos(x)}{x} - \sin(x) \ln(x) \right) + \frac{1}{x} \)
3. (20 pts) Consider the following graph of the function $f(t)$:

(a) $\lim_{t \to -2^-} f(t) = 3$

(b) $\lim_{t \to -6} f(t) = -4$

(c) $\lim_{t \to -8} f(t) = -6$

(d) Estimate the instantaneous rate of change of $f$ at $t = 10$ (don’t worry too much about getting it perfect).

Either draw in the tangent line at $t = 10$ or the secant line on $[10, 11]$ or $[9, 10]$. This gives us an estimate between -2 and -3.

(e) Estimate the average rate of change of $f$ on the interval $[2, 6]$. $\frac{f(6) - f(2)}{6 - 2} = \frac{5 - 3}{4} = \frac{2}{4} = \frac{1}{2}$

(f) Does the graph $f$ have any vertical asymptotes? If so, what are the equations? Yes, there’s a vertical asymptote at $t = -2$. We get a vertical asymptote at $t = c$ if either $\lim_{t \to c^-} f(t) = \pm \infty$ or $\lim_{t \to c^+} f(t) = \pm \infty$.

(g) An interval on which $f$ is decreasing is: $[-12, -8)$ or $(-2, 0]$ or $[2.5, 6)$ or $[7, 12]$ or some interval subset of one of these intervals.

(h) Is $f$ a 1-to-1 function, and why?

No it isn’t 1-1 function. It doesn’t satisfy the horizontal line test. For example, $f(-11) = f(-4) = 0$. 

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(i) The range of $f$ on the interval $[-12, 12]$ is: $(-6, \infty)$

(j) Put an $x$ on the $t$-axis at every point in the domain of $f$ where $f$ is not continuous.
   Not continuous at $t = -12, -8, -2, 6, 12$.

(k) Put a circle $\circ$ on the $t$-axis at every point in the domain of $f$ where $f$ is continuous but not differentiable.
   The function $f$ happens to be differentiable whenever it is continuous (though this is not true in general).

4. Find the equation of the tangent line to the curve

$$x + \sqrt{xy} = 6$$

at the point $(4, 1)$.

**Sol:** Since $x + \sqrt{xy} = 6$, differentiate both sides with respect to $x$ we get

$$1 + \frac{1}{2\sqrt{xy}}(y + xy') = 0$$

$$\Rightarrow \frac{x}{2\sqrt{xy}}y' = -1 - \frac{y}{2\sqrt{xy}}$$

$$\Rightarrow y' = \frac{2\sqrt{xy}}{x}(-1 - \frac{y}{2\sqrt{xy}})$$

$$= \frac{2\sqrt{xy}}{x} - \frac{y}{x}$$

Therefore at the point $(4, 1)\ y' = -\frac{4}{4} - \frac{1}{4} = -\frac{5}{4}$.
So the equation of the tangent line at the point $(4, 1)$ is

$$y - 1 = -\frac{5}{4}(x - 4)$$

*i.e.*

$$y = -\frac{5}{4}x + 6$$
5. Suppose $f(x)$ is a continuous function with domain $[-5, 5]$. Let $g(x) = x$. If $f(-5) = 6$ and $f(5) = -6$, show that there is $c \in [-5, 5]$ such that $f(c) = g(c)$.

**Pf:** Let $h(x) = f(x) - g(x)$. Since $f(x)$ and $g(x)$ are continuous functions, $h(x)$ is also continuous on $[-5, 5]$.

\[
h(-5) = f(-5) - g(-5) = 6 - (-5) = 11 > 0
\]

\[
h(5) = f(5) - g(5) = -6 - 5 = -11 < 0
\]

By *Intermediate Value Theorem*, there exists a $c \in (-5, 5)$ such that $h(c) = 0$. i.e., $f(c) - g(c) = 0$, i.e., $f(c) = g(c)$. 

6. Consider the function
\[ f(x) = \begin{cases} 
  x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\
  0 & x = 0 
\end{cases} \]

(a) Is \( f \) differentiable at \( x = 3 \), and why or why not?

\textit{Sol:} Since \( \frac{1}{x} \) is differentiable at \( x = 3 \) and \( \sin(x) \) is differentiable at \( x = \frac{1}{3} \), then \( \sin\left(\frac{1}{x}\right) \) is differentiable at \( x = 3 \). \( x^2 \) is also differentiable at \( x = 3 \), so by product rule, \( x^2 \sin\left(\frac{1}{x}\right) \) is differentiable at \( x = 3 \), i.e., \( f(x) \) is differentiable at \( x = 3 \).

(b) Calculate the derivative of \( f \) at \( x = 0 \).

\textit{Sol:} By definition
\[ f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) \]
Define \( q(h) = h \sin\left(\frac{1}{h}\right) \), \( p(h) = -|h| \), \( r(h) = |h| \), since \(-1 \leq \sin\left(\frac{1}{h}\right) \leq 1\), we have
\[ p(h) \leq q(h) \leq r(h) \]
Since \( \lim_{h \to 0} p(h) = \lim_{h \to 0} r(h) = 0 \), by \textit{Squeeze Theorem}, \( \lim_{h \to 0} q(x) = 0 \), i.e.,
\[ \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \]
Therefore \( f'(0) = 0 \).

(c) Decide if \( f(x) \) is even, odd, both, or neither.

\textit{Sol:} If \( x \neq 0 \), \( f(-x) = (-x)^2 \sin\left(\frac{1}{-x}\right) = x^2 \sin\left(-\frac{1}{x}\right) = -x^2 \sin\left(\frac{1}{x}\right) = -f(x) \).
If \( x = 0 \), \( f(-0) = f(0) = 0 = -f(0) \).
So \( f(x) \) is an odd function.