Introduction to Thompson’s group

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Abstract

Forty years ago Richard Thompson introduced a fascinating discrete group $F$, which has become a test case for many questions in geometric group theory. I will describe $F$ from several different points of view and state some known results and open problems.
1. Thompson’s definition

\[ F < T < V \]

\( F \) is the group of associative laws, \( T \) allows cyclic rearrangements, \( V \) allows arbitrary rearrangements. We only consider \( F \).

\[
\begin{align*}
    x_0 &: a(bc) \rightarrow (ab)c \\
    x_1 &: a(b(cd)) \rightarrow a((bc)d) \\
    x_2 &: a(b(c(de))) \rightarrow a(b((cd)e))
\end{align*}
\]

Expansion: Replace \( a, b, c, \ldots \) by expressions.

\[
\begin{align*}
    A(BC) &\rightarrow (AB)C' \\
    \overline{a(b(c(de)))} &\rightarrow \overline{(ab)(c(de))} \\
    \overline{A(B(CD))} &\rightarrow A((BC)D) \\
    \overline{(ab)(c(de))} &\rightarrow \overline{(ab)((cd)e)}
\end{align*}
\]
Composition

\[
\begin{align*}
    \mathcal{F} & \rightarrow (ab)(c(de)) \\
    \mathcal{F} & \rightarrow (ab)((cd)e)
\end{align*}
\]

A relation:

\[
x_1x_0 = x_0x_2 \quad \text{or} \quad x_1^{x_0} = x_2
\]

More generally,

\[
x_nx_i = x_i x_{n+1} \quad \text{or} \quad x_i^{x_n} = x_{n+1} \quad (i < n)
\]

Fact: \(x_0, x_1, x_2, \ldots\) generate \(\mathcal{F}\), and these are defining relations.
2. Combinatorial group theory

\[ F = \langle x_0, x_1, x_2, \ldots ; x_n^x = x_{n+1} \text{ for } i < n \rangle \]

\[ x_n x_i \rightarrow x_i x_{n+1} \quad \text{(smaller subscripts first)} \]

\[ x_i^{-1} x_n \rightarrow x_{n+1} x_i^{-1} \quad \text{(positive before negative)} \]

\[ x_n^{-1} x_i \rightarrow x_i x_{n+1}^{-1} \quad \text{(positive before negative)} \]

\[ x_i^{-1} x_n^{-1} \rightarrow x_{n+1}^{-1} x_i^{-1} \quad \text{(smaller subscripts last)} \]

Normal forms:

\[ f = x_{i_1} x_{i_2} \cdots x_{i_k} x_{j_l}^{-1} \cdots x_{j_2}^{-1} x_{j_1}^{-1} \quad (i_1 \leq \cdots \leq i_k, \ j_1 \leq \cdots \leq j_l) \]

Unique if reduced: If \( x_i \) and \( x_i^{-1} \) both occur, then so does \( x_{i+1} \) or \( x_{i+1}^{-1} \).

\[ x_0 x_1 x_1 x_3 x_5^{-1} x_4^{-1} x_1^{-1} x_0^{-1} = x_0 x_1 x_2 x_4^{-1} x_3^{-1} x_0^{-1} \]
3. Group of fractions

$F$ is the group of right fractions of its positive semigroup $P$:

$$f \in F \implies f = pq^{-1} \quad (p, q \in P)$$

$P$ has a concrete interpretation as the semigroup of binary forests (Belk, Brin).
Relations

\[ x_1x_0 = x_0x_2 \]

\[ x_2x_0 = x_0x_3 \]
4. Dyadic PL-homeomorphisms of $I$ (or $\mathbb{R}_+$ or $\mathbb{R}$)

$F \cong \text{PL}_2(I) \cong \text{PL}_2(\mathbb{R}_+) \cong \text{PL}_2(\mathbb{R})$. All slopes are integral powers of 2, all breakpoints have dyadic rational coordinates, integer translation near $\pm \infty$ if use $\mathbb{R}_+$ or $\mathbb{R}$. 
$\text{PL}_2(I) \cong \text{PL}_2(\mathbb{R}_+) \cong \text{PL}_2(\mathbb{R})$
5. Tree and forest diagrams

Binary trees encode binary subdivisions or parenthesized expressions.

If use $\mathbb{R}_+$, get forest diagrams (but we knew this already).

If use $\mathbb{R}$, get doubly-infinite forest diagrams.
6. Universal conjugacy idempotent

(Freyd–Heller, Dydak) $F$ is the universal example of a group with an endomorphism that is idempotent up to conjugacy:

\[ \phi(x_n) = x_{n+1}, \quad \phi^2 = \phi^{x_0} \]

Homeomorphism interpretation: $\phi(f) = "f"$ concentrated on $[1/2, 1]$.

Universality: Given any $\phi: G \to G$ with $\phi^2$ conjugate to $\phi$, need $x_0$ so that

(1) \[ \phi^2 = \phi^{x_0}, \]

then need $x_1 = \phi(x_0), x_2 = \phi(x_1), \ldots$. Equation (1) forces

\[ x_{n+1} = x_n^{x_0} \quad (n > 0), \]

apply $\phi$ to get remaining relations.
7. Algebra automorphisms

(Galvin–Thompson) $F$ is isomorphic to the group of order-preserving automorphisms of a free Cantor algebra:

$$\mu : X \times X \to X$$  \hspace{1cm} \text{(bijection)}

Everything splits uniquely as a product.

$$a = a_0 a_1 = a_0 (a_{10} a_{11}) \quad a = a_0 a_1 = (a_{00} a_{01}) a_1$$

Every tree diagram (or associative law) gives an automorphism.
Why is $F$ interesting?

- Comes up in many ways.
- Has interesting properties.
- Almost every question is a challenge.
Known properties of $F$

1. Good finiteness properties: Two generators $x_0, x_1$. Two relations
\[ x_1^{x_0} x_0 = x_1^{x_0} x_1 \] and
\[ x_1^{x_0} x_0 x_0 = x_1^{x_0} x_0 x_1. \] And so on (Brown–Geoghegan).

2. $F$ is “almost simple”.

\[ 1 \to F' \to F \to \mathbb{Z} \times \mathbb{Z} \to 0 \]

3. Although highly nonabelian, $F$ admits a product $F \times F \to F$, associative up to conjugacy. [No identity; $1 \ast 1 = 1$, but $1 \ast f = \phi(f)$ in general.]

4. $F$ has no free subgroups.

5. $F$ is not elementary amenable.

6. Isoperimetric constant with respect to $x_0, x_1$ is $\leq 1/2$ (Belk–Brown).

7. The Poisson boundary for (some) symmetric random walks is nontrivial (Kaimanovich).

8. Homology is known (B–G): $H_n(F') \cong \mathbb{Z} \oplus \mathbb{Z}$ for all $n \geq 1$. 
9. Homology and cohomology are known as rings (B): \( H_*(F) \) is an associative algebra (without identity) generated by \( e \) (degree 0), \( \alpha, \beta \) (degree 1), subject to relations

\[
e^2 = e \\
e\alpha = \beta e = 0 \\
\alpha e = \alpha \ , \quad e\beta = \beta
\]

Consequence: \( \alpha^2 = \beta^2 = 0 \), alternating products \( \alpha\beta\alpha \cdots \) and \( \beta\alpha\beta \cdots \) give basis in positive dimensions. \( H_*(F) \cong \wedge(a, b) \otimes \Gamma(u) \).

10. \( F \) is orderable.

11. Easy algorithm for computing length function (Fordham, Belk–Brown).

12. Growth series explicitly known for \( P \) (Burillo, B–B):

\[
p(x) = \frac{1 - x^2}{1 - 2x - x^2 + x^3}
\]
Open problems

1. Is $F$ amenable?

2. Is the isoperimetric constant $1/2$? Is it $\geq 1/2$ for all generating sets?

3. Describe the Poisson boundary or other invariants of random walk.

4. Is $F$ automatic?

5. What is the exponential growth rate of $F$?

6. Is the growth series rational?

7. Is it true that every subgroup of $F$ is either elementary amenable or contains an isomorphic copy of $F$?

See

http://www.aimath.org/WWN/

for more (to appear).