$FP_\infty$ GROUPS AND HNN EXTENSIONS

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A group $G$ is said to be of type $FP_\infty$ if the $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective resolution $(P_i)$ of finite type (i.e., with each $P_i$ finitely generated). If $G$ is finitely presented, this is equivalent by Wall [5, 6] to the existence of an Eilenberg-Mac Lane complex $K(G,1)$ of finite type (i.e., with finitely many cells in every dimension). Up to now, all known torsion-free groups of type $FP_\infty$ have had finite cohomological dimension; in fact, they have admitted a finite $K(G,1)$-complex. We announce here the first known example of a torsion-free group of type $FP_\infty$ with infinite cohomological dimension. This solves Wall’s problem F11 [7]. We show in addition that our group, which we denote by $F$, satisfies $H^n(F,\mathbb{Z}F) = 0$ for all $n$. As far as we know, $F$ is also the first example of an $FP_\infty$ group with this property. The vanishing of $H^*(F,\mathbb{Z}F)$ is a consequence of results of independent interest concerning the cohomology of HNN extensions (or, more generally, fundamental groups of graphs of groups) with free coefficients.

The group $F$ is defined by the presentation $\langle x_0, x_1, x_2, \ldots; x_i^{-1}x_n x_i = x_{n+1} \text{ for } i < n \rangle$. It has previously arisen in two contexts: (i) finitely presented infinite simple groups (R. J. Thompson [unpublished]); and (ii) unsplittable free-homotopy idempotents (Freyd and Heller [3], Dydak and Minc [2]). $F$ was previously known to be finitely presented, torsion-free, and of infinite cohomological dimension. (In fact, $F$ has a subgroup which is free abelian of infinite rank.) Our contribution, therefore, is

**Theorem 1.** The group $F$ described above is of type $FP_\infty$ and satisfies $H^*(F,\mathbb{Z}F) = 0$.

The proof that $F$ is of type $FP_\infty$ goes as follows. Let $\phi: F \to F$ be the shift map, $\phi(x_i) = x_{i+1}$. Note that $\phi^2 = T_{x_0} \circ \phi$, where $T_{x_0}$ is the conjugation map $x \mapsto x_0^{-1}xx_0$; thus $\phi$ is idempotent up to conjugacy. We construct the universal example of a semicubical complex $K$ with (a) a free right $F$-action; (b) a basepoint-preserving cubical endomorphism $\psi: K \to K$ compatible with $\phi$; and (c) a homotopy from $\psi^2$ to $\rho_{x_0} \circ \psi$ compatible with $\phi^2$, where $\rho_{x_0}(e) = ex_0$. (The motivation for this comes from (ii) above; $K$ should be thought of as the universal cover of a complex with a free-homotopy idempotent, and $\psi$ should be thought of as a lift of the idempotent to $K$.) We prove by a direct combinatorial argument that $K$ is acyclic; the chain complex $C$ of $K$ is therefore a free resolution of $\mathbb{Z}$ over $\mathbb{Z}F$. Unfortunately, $C$ is not of finite type. But we are able to find a contractible chain subcomplex $D \subset C$ such

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Received by the editors April 19, 1983.

1980 Mathematics Subject Classification. Primary 20J05; Secondary 20E06, 18G20, 55P20.

1Authors partially supported by the National Science Foundation.
that the quotient \( P = C/D \) is free of finite rank in every dimension. This is then the desired finite type resolution.

One can give a direct description of this resolution \( P \). It is free of rank 1 in dimension 0 and free of rank 2 in every positive dimension. Moreover, there are formulas for computing the boundary operator inductively. In spite of this explicit description of \( P \), however, we know of no proof of its acyclicity other than the one outlined above which uses the "big" resolution \( C \).

We turn now to the assertion in Theorem 1 that \( H^*(F, \mathbb{Z}F) = 0 \). Let \( F_1 \) be the subgroup of \( F \) generated by the \( x_i \) for \( i \geq 1 \). It is known that \( F_1 \) is isomorphic to \( F \) via the shift map \( \phi \) and that \( F \) is the HNN extension of \( F_1 \) with respect to the monomorphism \( \phi | F_1 \) (with \( x_0 \) as the stable letter).

We now appeal to a general result about HNN extensions in which the base group and associated subgroups satisfy appropriate finiteness conditions; for simplicity, we will state a special case of this result which suffices for the present application.

**Theorem 2.** Let \( G \) be an HNN extension in which the base group \( G_1 \) and associated subgroups \( A \) and \( B \) are of type \( FP_\infty \). Assume that one of the associated subgroups, say \( A \), has the property that the restriction map \( H^*(G_1, \mathbb{Z}G_1) \to H^*(A, \mathbb{Z}G_1) \) is a monomorphism. Then in the Mayer-Vietoris sequence

\[
\cdots \to H^q(G, \mathbb{Z}G) \to H^q(G_1, \mathbb{Z}G) \overset{\alpha}{\to} H^q(A, \mathbb{Z}G) \to \cdots
\]

the map \( \alpha \) is a monomorphism.

This generalizes a result of Bieri [1, Theorem 6.6], in which \( A \) and \( B \) were both assumed to be of finite index in \( G_1 \). Note that our hypothesis about the restriction map holds whenever one of these subgroups is of finite index. In particular, it holds when \( G_1 = A \), which is the case in our present application with \( G = F \) and \( G_1 = A = F_1 \). If we now assume inductively that \( H^{q-1}(F, \mathbb{Z}F) = 0 \), it follows that \( H^{q-1}(F, L) = 0 \) for any free \( \mathbb{Z}F \)-module \( L \).

Since \( F_1 \approx F \), this yields \( H^{q-1}(F_1, \mathbb{Z}F) = 0 \), so the Mayer-Vietoris sequence takes the form

\[
0 \to H^q(F, \mathbb{Z}F) \to H^q(F_1, \mathbb{Z}F) \overset{\alpha}{\to} H^q(F_1, \mathbb{Z}F) \to \cdots.
\]

Theorem 2 now implies that \( H^q(F, \mathbb{Z}F) = 0 \), as required. This completes the sketch of the proof of Theorem 1.

To prove Theorem 2, one can give a normal form argument. Alternatively, there is a proof which makes use of the tree associated to the HNN extension [4]. This second proof is of interest because it leads to a generalization of Theorem 2 to fundamental groups of graphs of groups, as follows.

Let \( G \) be the fundamental group of a finite graph of groups [4]. We will assume for simplicity that the vertex and edge groups are all of type \( FP_\infty \), although this hypothesis can be weakened. Let \( X \) be the associated tree. For each integer \( q \) there is a "coefficient system" \( D^q \) on \( X \) which associates to each vertex or edge \( \sigma \) of \( X \) the group \( H^q(G_{\sigma}, \mathbb{Z}G_{\sigma}) \), where \( G_{\sigma} \) is the isotropy subgroup of \( G \) at \( \sigma \), and which associates to each incidence relation "\( \nu \) is a
vertex of $e''$ the map $H^q(G_v, \mathbb{Z}G_v) \to H^q(G_e, \mathbb{Z}G_e)$ induced by the inclusion $G_e \hookrightarrow G_v$ and the canonical projection $\mathbb{Z}G_v \to \mathbb{Z}G_e$. Our hypotheses imply that this coefficient system is locally finite in a suitable sense, so that we can form the complex $C_*^c(X, \mathcal{D}^q)$ of cochains with compact supports and hence the cohomology groups $H_*^c(X, \mathcal{D}^q)$. One now verifies that in the Mayer-Vietoris sequence with $\mathbb{Z}G$-coefficients, the map analogous to the map $\alpha$ of Theorem 2 can be identified with the coboundary map $C_*^c(X, \mathcal{D}^q) \to C_*^{c+1}(X, \mathcal{D}^q)$. The content of Theorem 2, then, is that (under the hypotheses of the latter) $H_*^0(X, \mathcal{D}^q) = 0$. It is not hard to verify this by directly checking definitions. In the general case this discussion yields

**Theorem 3.** Let $G$ be the fundamental group of a finite graph of groups of type $FP_\infty$ as above. Then there is a short exact sequence

$$0 \to H_*^1(X, \mathcal{D}^{q-1}) \to H^q(G, \mathbb{Z}G) \to H_*^0(X, \mathcal{D}^q) \to 0.$$

In particular, this shows that $H^q(G, \mathbb{Z}G) \approx H_*^1(X, \mathcal{D}^{q-1})$ under the hypotheses of Theorem 2. We will give further generalizations and applications of Theorem 3 elsewhere.

**References**


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