P1 Any unlabeled tree on 4 vertices is either

Figure 1: $A_4$

or

Figure 2: $D_4$

Note that there are 4 labeled trees of type $D_4$, as after choosing the middle vertex, the rest are indistinguishable. For $A_4$, there are 4! labelings, but each permutation corresponds to the same labeled tree as its reverse (by rotating the graph in the plane by 180°). So there are 4 labeled trees of type $D_4$, and 4!/2 = 12 of type $A_4$. Labeled trees correspond bijectively to Prüfer codes, and we’ll write down the Prüfer code for this labeled tree:

Figure 3: $D_4$

Recall that to find the Prüfer code of a given labeled tree, we first remove the lowest labeled leaf (in this case, 2) and record which vertex it’s adjacent to (in this case, 1), then repeat this process, (this time with the leaf 3 adjacent to 1) until we get a tree on 2 vertices, which is unique. So the Prüfer code of the above tree is {1, 1}. 
P2 Recall that the number of closed walks of length \( l \) in \( G \) is equal to 
\[
\sum_{i=1}^{21} (\lambda_i)^l
\]
where the \( \lambda_i \) are the eigenvalues of \( A \), where \( A \) is the adjacency matrix of \( G \). From the given formula we deduce that these eigenvalues are 3 (once), 5 (4 times), 8 (once), 1 (7 times) and 0 (8 times). Note that by adding 3 loops at each vertex to form \( G' \), its adjacency matrix \( A' \) is equal to \( A + 3 \cdot I \). Since the \( \lambda_i \)'s are precisely the solutions to 
\[
\det(A - \mu I) = 0,
\]
it is clear that the solutions to 
\[
\det(A' - \mu I) = \det(A - 3 \cdot I - \mu I) = 0,
\]
i.e. the eigenvalues of \( A' \) are \( \lambda_i + 3 \). Applying the theorem again now, the number of closed walks of length \( l \) in \( G' \) is 
\[
6^l + 4 \cdot 8^l + 11^l + 7 \cdot 4^l + 8 \cdot 3^l.
\]

P3 Let \( P^\circ(n,k) \) be the number of partitions of \([n]\) into \( k \) ordered nonempty subsets. Note that if \( k > n \), then \( P^\circ(n,k) = 0 \).

Note that the number of ways that \([n]\) can be partitioned into \( k \) ordered subsets, out of which the subsets indexed by \( i_1, \ldots, i_{k-j} \) are empty is 
\[
j^n.
\]
Indeed, we can place any of the \( n \) elements in any of the \( j \) subsets that are not prescribed to be empty.

Since we want to partition \([n]\) into \( k \) nonempty ordered subsets, we’ll use inclusion-exclusion, noting that there are \( \binom{k}{k-j} = \binom{k}{j} \) ways to pick which \( k-j \) sets are required to be empty.

\[
P^\circ(n,k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.
\]
Recall that the Stirling numbers of the second kind \( S(n,k) \) are the number of ways of partitioning \([n]\) into \( k \) unordered subsets. Therefore
\[
S(n,k) = \frac{1}{k!} P^\circ(n,k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n.
\]

P4 Note that the formula is equivalent to 
\[
\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k}.
\]
We will give a bijection between odd and even size subsets of \([n]\). Define the map \( \Phi \) from subsets of \([n]\) to subsets of \([n]\) by:
\[
\Phi(S) = \begin{cases} 
S \cup \{n\} & \text{if } n \notin S \\
S \setminus \{n\} & \text{if } n \in S
\end{cases}
\]
Notice that if we restrict \( S \) to sets of even cardinality, we get sets of odd cardinality. Also, \( S \) is easily seen to be bijective.

P5 Consider the element 1. It can be paired with \( 2n - 1 \) possible elements. After pairing it, consider the element which is the smallest among those that are not yet paired. It can be paired with \( 2n - 3 \) possible elements. After pairing it, consider the element which is the smallest among those that are not yet paired. It can be paired with \( 2n - 5 \) possible elements. Continue in this fashion to arrive to 
\[
1 \cdot 3 \cdot \cdots (2n-3) \cdot (2n-1).
\]