1. (a) Let the contour $\Gamma$ go from $z = 2$ to $z = -2$ counterclockwise around the upper semicircle $\{ |z| = 2, \text{Im}(z) \geq 0 \}$. By parametrizing $\Gamma$, compute $\int_{\Gamma} \frac{1}{z} dz$.

(b) Let $L(z)$ be the branch of $\log(z)$ that takes the argument to be in the interval $(-\pi/2, 3\pi/2)$. Use $L(z)$ and the Fundamental Theorem of Calculus to compute the integral from part (a) by a different method.

2. Provide an example of an analytic function $f$ with a pole of order 4 at $z_0 = 2i$ such that the residue $\text{Res}(f; 2i) = -3$.

3. Let $f$ be analytic on the entire complex plane except at isolated singularities $z_1, \ldots, z_k$. Prove that $f$ has an antiderivative if and only if $\text{Res}(f; z_j) = 0$ for every $1 \leq j \leq k$.

4. Let $C$ be the unit circle oriented counterclockwise, and define $f(z) = \int_{C} \frac{\sin(w)}{w-z} dw$, $g(z) = \int_{C} \frac{\sin(w)}{(w-z)^2} dw$ for $z$ not on the circle. Compute: $f(\pi/6), f(\pi/3), g(\pi/6), g(\pi/3)$.

5. Recall the Cauchy estimates: If $f$ is analytic on and inside the circle $C_R$ of radius $R$ centered at $z_0$, and $|f(z)| \leq M$ for all $z$ on $C_R$, then for all $n \geq 0$,
   $$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$ Suppose $f$ is an entire function such that $|f(z)| \leq |z|^2$ for all $z \in \mathbb{C}$. Use the Cauchy estimates to prove that $f$ must be a polynomial of degree at most 2, that is, $f(z) = c_0 + c_1 z + c_2 z^2$, and furthermore that $|c_2| \leq 1$. Hint: Consider the Taylor series for $f$ centered at the origin.

6. Compute the Laurent series for $f(z) = \frac{2}{z-1} + \frac{3}{z+5}$ in the annulus $\{ 2 < |z| < 3 \}$. What is the largest annulus $\{ r < |z| < R \}$ on which the Laurent series converges?
7. Find the singularities of \( f(z) = \frac{e^{1/z} [\cos(z) - 1]}{(z + \pi)^2 (z + 2\pi)^4} \) and classify them as removable, essential, or poles. Find the order of each pole.

8. What is the radius of convergence for the Taylor series of \( f(z) = \frac{e^{\cos(z)}}{z^2 + 9} \) centered at \(-4\)?

9. Suppose that \( g_1(z) \) and \( g_2(z) \) are both analytic at \( z_0 \). Also assume that \( g_1(z_0) \neq 0 \), while \( g_2 \) has a simple zero at \( z_0 \), so the Taylor series are

\[
\begin{align*}
g_1(z) &= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad \text{with } a_0 \neq 0, \\
g_2(z) &= b_1(z - z_0) + b_2(z - z_0)^2 + \cdots \quad \text{with } b_1 \neq 0.
\end{align*}
\]

Prove that \( f(z) = \frac{g_1(z)}{g_2(z)} \) has a simple pole at \( z_0 \), and that

\[
\text{Res}(f; z_0) = \frac{a_0}{b_1} = \frac{g_1(z_0)}{g'_2(z_0)}.
\]

10. Use residue theory to compute \( \int_0^{2\pi} \frac{1}{13 + 12 \cos \theta} \, d\theta \).  

*Hint:* \((2z + 3)(3z + 2) = 6z^2 + 13z + 6\).

11. Compute the residue of \( f(z) = \frac{1}{z^2 + 2z^3} \) at \( z = 0 \).

12. Use residue theory to compute p.v. \( \int_{-\infty}^{\infty} \frac{\sin(2x)}{x^2 + 1} \, dx \).