1. Find all points on the graph of the function \( f(x) = 2 \cos x - \sin^2 x \) at which the tangent line is horizontal. (Note: \( \sin^2 x = (\sin x)^2 \).)

We use the chain rule to get
\[
\frac{d}{dx}(f(x)) = -2 \sin x - 2 \sin x \cos x.
\]

If a point on the graph of \( f(x) \) has a horizontal tangent line, then \( f'(x) = 0 \) at the \( x \)-value corresponding to that point. Setting \( f'(0) = 0 \) we have
\[
0 = -2 \sin x - 2 \sin x \cos x = -2 \sin x(1 + \cos x)
\]

So if \(-2 \sin x = 0\) or if \(1 + \cos x = 0\) then we have a horizontal tangent at that \( x \)-value.

\[-2 \sin x = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \ n \text{ an integer.}\]

\[1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow x = \pi + 2n\pi, \ n \text{ an integer}\]

(these are odd multiples of \( \pi \)).

Since all of the integer multiples of \( \pi \) include all of the odd multiples of \( \pi \), \( f(x) \) has a horizontal tangent at all \( x = n\pi \) where \( n \) is an integer. This corresponds to the points \((2n\pi, 2)\) and \(((2n+1)\pi, -2)\).

2. Consider the function \( g(x) = |x|\sqrt{2-x^2} \).

(a) What is the domain of \( g(x) \)?

\( g(x) \) has domain all \( x \)-values where \( 2 - x^2 \geq 0 \) since we can only take the square root of positive numbers. So
\[
2 - x^2 \geq 0 \iff 2 \geq x^2 \iff -\sqrt{2} \leq x \leq \sqrt{2}.
\]

So the domain of \( g(x) \) is the interval \([-\sqrt{2}, \sqrt{2}]\).

(b) Find an equation of the tangent line to the curve \( y = g(x) \) at the point \((1, 1)\).

Since \( 0 \leq 1 \leq \sqrt{2} \), we can get rid of the absolute value sign and rewrite \( g(x) \) as
\[
g(x) = x\sqrt{2-x^2} \text{ when } 0 \leq x \leq \sqrt{2}.
\]

So for \( 0 < x < \sqrt{2} \),
\[
g'(x) = \frac{1}{2}(2-x^2)^{-1/2} \cdot (-2x) + \sqrt{2-x^2} = \frac{-x^2}{\sqrt{2-x^2}} + \sqrt{2-x^2}.
\]

The tangent line to the curve \( y = g(x) \) at the point \((1, 1)\) has slope \( g'(1) = 0 \). In point-slope form the equation of the tangent line to \( y = g(x) \) at \((1, 1)\) is
\[
y - 1 = 0(x - 1) \Rightarrow y = 1.
\]
(c) Does \( g'(0) \) exist? (Show your work.)

When \(-\sqrt{2} \leq x \leq 0 \) or \( 0 \leq x \leq \sqrt{2} \), we can write down a nice equation for \( g'(x) \) (see (a) for the second case). So we need to see whether or not \( g'(x) \) matches up on the left and right of \( x = 0 \). There are two ways to think of this: if we take the limit of the slope of the secant lines on the left of \( x = 0 \) and the right of \( x = 0 \), they must equal each other for \( g'(x) \) to exist, or if we take the limits of the function \( g'(0) \) as \( x \to 0 \) from the right and the left, they must equal each other for \( g'(0) \) to exist.

Method one:

\[
\lim_{h \to 0^+} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0^+} \frac{h\sqrt{2 - h^2} - 0}{h} = \sqrt{2}
\]

\[
\lim_{h \to 0^-} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0^-} \frac{-h\sqrt{2 - h^2} - 0}{h} = -\sqrt{2}
\]

Since the left and right hand limits of the slopes of secant lines through \((0, g(0))\) aren’t equal, this means that \( g'(0) \) DNE.

Method two:

\[
\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} \left[ \frac{-x^2}{\sqrt{2 - x^2}} + \sqrt{2 - x^2} \right] = \sqrt{2}
\]

\[
\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} \left[ \frac{x^2}{\sqrt{2 - x^2}} - \sqrt{2 - x^2} \right] = -\sqrt{2}
\]

Since \( g'(x) \) isn’t continuous at \( x = 0 \), this means that \( g'(0) \) DNE.

3. Suppose that

\[
\frac{d}{dx} [f(2x)] = 12x.
\]

Find \( f'(x) \).

We use the chain rule first to calculate \( \frac{d}{dx} [f(2x)] = f'(2x) \cdot 2 \). So

\[
= f'(2x) \cdot 2 = 12x,
\]

and

\[
f'(2x) = 6x = 3(2x).
\]

We make the substitution \( u = 2x \) and get \( f'(u) = 3u \), so \( f'(x) = 3x \).