1. Show that the equation $\cos x = x^3$ has at least one real root.
(Note: (1) A root of an equation is an $x$ value that makes the equation true). (2) You do not need to find a root, just show one exists.)

This problem should scream *Intermediate Value Problem*. Since we aren’t interested in finding an exact $x$-value that solves $\cos x = x^3$ but rather want to know if a solution exists, we want to use the Intermediate Value Theorem (IVT). To use the IVT we can only consider the question of whether a function equals a number, so we investigate whether or not the equation $\cos x - x^3 = 0$ has any real roots. If we graph the function $f(x) = \cos x - x^3$ on our graphing calculators, we see that there is a root for some $x$-value between 0 and 2 (this is the same as saying the graph of $f(x)$ has an $x$-intercept between 0 and 2). We will try to use the IVT for $f(x) = \cos x - x^3$ on the interval $[0, 2]$ with $N = 0$. Since $\cos x$ and $x^3$ are continuous everywhere, by theorem #4 in section 2.4 we have that $f(x) = \cos x - x^3$ is continuous everywhere, and in particular $f(x)$ is continuous on $[0, 2]$. Also, $f(0) = \cos(0) - 0^3 = 1 > 0$ and $f(2) = \cos(2) - 2^3 < 0$ (here $-1 \leq \cos(2) \leq 1$ and $2^3 = 8$, so we definitely have a negative number). So for emphasis, $f(2) < 0 < f(0)$. By the IVT, there is a number $0 < c < 2$ with $f(c) = \cos(c) - c^3 = 0$. Or, there is a number $0 < c < 2$ with $\cos(c) = c^3$.

2. Suppose $f(x) = \frac{2}{x+3}$.

(a) Use the definition of limit to find a formula for $f'(a)$.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{(a+h)+3} - \frac{2}{a+3}}{h}$$

$$= \lim_{h \to 0} \frac{2(a+3) - 2(a+h+3)}{(a+h+3)(a+3) h}$$

$$= \lim_{h \to 0} \frac{2a + 6 - 2a - 2h - 6}{h(a + h + 3)(a + 3)}$$

$$= \lim_{h \to 0} \frac{-2h}{h(a + h + 3)(a + 3)}$$

$$= \lim_{h \to 0} \frac{-2}{h(a + h + 3)(a + 3)}$$

$$= \frac{-2}{(a + 0 + 3)(a + 3)}$$

$$= \frac{-2}{(a + 3)^2}$$

(b) Use part (a) to find an equation for the line tangent to $y = f(x)$ at the point $(a, f(a))$ when $a = 4$.

When $a = 4$, $(a, f(a)) = (4, f(4)) = (4, \frac{2}{7})$. The line tangent to $y = f(x)$ at this point has slope $f'(4) = -\frac{2}{49}$. An equation for this tangent line, in point-slope form, is

$$y - \frac{2}{7} = -\frac{2}{49}(x - 4).$$
3. Below is the graph of $f(x)$. In the space provided, sketch a graph of the function $y = f'(x)$.

The graph of $f'(x)$ should look something like this: