To fill in the missing step in this argument we will need a technical lemma about deforming maps to create some linearity. Define a **polyhedron** in \( \mathbb{R}^n \) to be a subspace that is the union of finitely many convex polyhedra, each of which is a compact set obtained by intersecting finitely many half-spaces defined by linear inequalities of the form \( \sum_i a_i x_i \leq b \). By a **PL (piecewise linear) map** from a polyhedron to \( \mathbb{R}^k \) we shall mean a map which is linear when restricted to each convex polyhedron in some decomposition of the polyhedron into convex polyhedra.

**Lemma 4.10.** Let \( f : I^n \to Z \) be a map, where \( Z \) is obtained from a subspace \( W \) by attaching a cell \( e^k \). Then \( f \) is homotopic rel \( f^{-1}(W) \) to a map \( f_1 \) for which there is a polyhedron \( K \subset I^n \) such that:

(a) \( f_1(K) \subset e^k \) and \( f_1|K \) is PL with respect to some identification of \( e^k \) with \( \mathbb{R}^k \).

(b) \( K \supset f_1^{-1}(U) \) for some nonempty open set \( U \) in \( e^k \).

Before proving the lemma, let us see how it finishes the proof of the cellular approximation theorem. Composing the given map \( f : X^{n-1} \cup e^n \to Y^k \) with a characteristic map \( I^n \to X \) for \( e^n \), we obtain a map \( f \) as in the lemma, with \( Z = Y^k \) and \( W = Y^k - e^k \). The homotopy given by the lemma is fixed on \( \partial I^n \), hence induces a homotopy \( f_1 \) of \( f | X^{n-1} \cup e^n \) fixed on \( X^{n-1} \). The image of the resulting map \( f_1 \) intersects the open set \( U \) in \( e^k \) in a set contained in the union of finitely many hyperplanes of dimension at most \( n \), so if \( n < k \) there will be points \( p \in U \) not in the image of \( f_1 \). \( \square \)

**Proof of 4.10:** Identifying \( e^k \) with \( \mathbb{R}^k \), let \( B_1, B_2 \subset e^k \) be the closed balls of radius 1 and 2 centered at the origin. Since \( f^{-1}(B_2) \) is closed and therefore compact in \( I^n \), it follows that \( f \) is uniformly continuous on \( f^{-1}(B_2) \). Thus there exists \( \epsilon > 0 \) such that \( |x - y| < \epsilon \) implies \( |f(x) - f(y)| < \frac{1}{2} \) for all \( x, y \in f^{-1}(B_2) \). Subdivide the interval \( I \) so that the induced subdivision of \( I^n \) into cubes has each cube lying in a ball of diameter less than \( \epsilon \). Let \( K_1 \) be the union of all the cubes meeting \( f^{-1}(B_1) \), and let \( K_2 \) be the union of all the cubes meeting \( K_1 \). We may assume \( \epsilon \) is chosen smaller than half the distance between the compact sets \( f^{-1}(B_1) \) and \( I^n - f^{-1}(\text{int}(B_2)) \), and then we will have \( K_2 \subset f^{-1}(B_2) \).
Now we subdivide all the cubes of $K_2$ into simplices. This can be done inductively. The boundary of each cube is a union of cubes of one lower dimension, so assuming these lower-dimensional cubes have already been subdivided into simplices, we obtain a subdivision of the cube itself by taking its center point as a new vertex and joining this by a cone to each simplex in the boundary of the cube.

Let $g: K_2 \to e^k = \mathbb{R}^k$ be the map that equals $f$ on all vertices of simplices of the subdivision and is linear on each simplex. Choose a map $\varphi: K_2 \to [0, 1]$ with $\varphi(\partial K_2) = 0$ and $\varphi(K_1) = 1$, for example a map which is linear on simplices and has the value 1 on vertices of $K_1$ and 0 on all other vertices. Define a homotopy $f_t: K_2 \to e^k$ by the formula $(1 - t\varphi)f + (t\varphi)g$. Thus $f_0 = f$ and $f_1 | K_1 = g | K_1$. Since $f_t$ is the constant homotopy on $\partial K_2$, we may extend $f_t$ to be the constant homotopy of $f$ on the rest of $I^n$.

The map $f_1$ takes the closure of $I^n - K_1$ to a compact set $C$ which, we claim, is disjoint from the centerpoint 0 of $B_1$ and hence from a neighborhood $U$ of 0. This will prove the lemma, with $K = K_1$, since we will then have $f_1^{-1}(U) \subset K_1$ with $f_1$ PL on $K_1$ where it is equal to $g$.

The verification of the claim has two steps:

1. On $I^n - K_2$ we have $f_1 = f$, and $f$ takes $I^n - K_2$ outside $B_1$ since $f^{-1}(B_1) \subset K_2$ by construction.

2. For a simplex $\sigma$ of $K_2$ not in $K_1$ we have $f(\sigma)$ contained in some ball $B_\sigma$ of radius $\frac{1}{2}$ by the choice of $\epsilon$ and the fact that $K_2 \subset f^{-1}(B_2)$. Since $f(\sigma) \subset B_\sigma$ and $B_\sigma$ is convex, we must have $g(\sigma) \subset B_\sigma$, hence also $f_t(\sigma) \subset B_\sigma$ for all $t$, and in particular $f_1(\sigma) \subset B_\sigma$. We know that $B_\sigma$ is not contained in $B_1$ since $\sigma$ contains points outside $K_1$ hence outside $f^{-1}(B_1)$. The radius of $B_\sigma$ is half that of $B_1$, so it follows that 0 is not in $B_\sigma$, and hence 0 is not in $f_1(\sigma)$. □

This revised version of Lemma 4.10 requires a small adjustment in the proof of Theorem 4.23 on page 361. The sentence beginning “By repeated applications” about two-thirds of the way down the page should be modified to:

By repeated applications of Lemma 4.10 we may homotope $f$, through maps $(I^i, \partial I^i, f^{-1}) \to (X, B_i, x_0)$, so that there are simplices $\Delta_{i+1}^m \subset e_{i+1}^m$ and $\Delta_{i+1}^n \subset e_{i+1}^n$ for which the preimages $f^{-1}(\Delta_{i+1}^m)$ and $f^{-1}(\Delta_{i+1}^n)$ are finite unions of convex polyhedra, on each of which $f$ is the restriction of a linear map from $\mathbb{R}^i$ to $\mathbb{R}^{m+1}$ or $\mathbb{R}^{n+1}$. We may assume these linear maps are surjections by choosing the simplices $\Delta_{i+1}^m$ and $\Delta_{i+1}^n$ to lie in the complement of the images of the nonsurjective linear maps.