1. Show that if $M$ is a smooth submanifold of a manifold $Q$ then a map $f : M \to N$ is smooth if and only if for each point $x \in M$ there is an open neighborhood $U$ of $x$ in $Q$ and a smooth map $U \to N$ that agrees with $f$ on $M \cap U$.

2. Given a connected manifold $M$ and two points $x, y \in M$ show that there exists a diffeomorphism $f : M \to M$ with $f(x) = y$, and with $f$ equal to the identity outside some compact set in $M$. More generally, given two sets $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ of $n$ distinct points in $M$, show there is a compactly supported diffeomorphism $f : M \to M$ with $f(x_i) = y_i$ for each $i$, provided that the dimension of $M$ is at least 2. [Updated to include this dimension condition.]

3. Let $p : E \to M$ be a smooth vector bundle. For a closed set $K$ in $M$, let $s : K \to E$ be a smooth section, so $ps$ is the identity map on $K$. Show that $s$ can be extended to a smooth section $M \to E$.

4. For a space $X$, let $\kappa(X)$ denote the set of path-components of $X$. Show that if $X$ is a Lie group (or more generally a topological group) then $\kappa(X)$ inherits a group structure from $X$ such that the natural projection $X \to \kappa(X)$ sending $x \in X$ to its path-component $[x] \in \kappa(X)$ is a homomorphism. Deduce from this that the path-component of $X$ containing the identity element of $X$ is a normal subgroup, and give a direct proof of this fact that does not use the group $\kappa(X)$.

5. Suppose that $G$ is a connected Lie group and $H$ is a subgroup that contains an open neighborhood of the identity element of $G$. Show that $H = G$.

6. (a) Show that a discrete normal subgroup $H$ of a connected Lie group $G$ must lie in the center of $G$. (Here “discrete” means discrete as a subspace of $G$, i.e., the subspace topology on $H$ is the discrete topology.)

(b) Show that if $G$ is a Lie group and $H$ is a finite normal subgroup then the quotient group $G/H$ is a Lie group. (Don’t forget to check that $G/H$ is Hausdorff.)

(c) Let $PGL_n(\mathbb{R}) = GL_n(\mathbb{R})/H$ where $H$ is the subgroup of matrices which are a scalar multiple of the identity matrix. Show that $PGL_n(\mathbb{R})$ is a Lie group of dimension $n^2 - 1$. [Hint: first show that $GL_n(\mathbb{R})$ can be replaced by the subgroup consisting of matrices of determinant $\pm 1$.]

(d) How many path-components does $PGL_n(\mathbb{R})$ have? Also, how does $PGL_n(\mathbb{R})$ compare with $PSL_n(\mathbb{R})$ where we start with $SL_n(\mathbb{R})$ instead of $GL_n(\mathbb{R})$?

(e) For each integer $k > 1$ find a normal subgroup of $U(n)$ which is cyclic of order $k$. 