1. Give details in the proof of the fact, stated in class, that for a composition \(gf\) of smooth maps \(f: M \to N\) and \(g: N \to P\), the induced maps on tangent bundles \(Df: TM \to TN\) and \(Dg: TN \to TP\) satisfy \(D(gf) = Dg \circ Df\). (Here we are using the definition of tangent bundles in terms of coordinate charts.)

2. Show that for smooth manifolds \(M\) and \(N\), \(T(M \times N)\) is diffeomorphic to \(TM \times TN\).

3. (a) Show that if \(p_1: E_1 \to M\) and \(p_2: E_2 \to M\) are smooth vector bundles, then so is their direct sum \(E_1 \oplus E_2 \to M\).

   (b) Verify that in a smooth vector bundle \(p: E \to M\) the operations of vector addition \(E \oplus E \to E, (v, w) \mapsto v + w\), and scalar multiplication \(\mathbb{R} \times E \to E, (t, v) \mapsto tv\), are smooth maps.

   (c) Verify that a smooth section of a smooth vector bundle is a smooth embedding.

4. (a) Show that the Klein bottle \(K = (S^1 \times [0, 1])/(z, 0) \sim (\bar{z}, 1)\) has a nonvanishing (i.e., nowhere zero) vector field.

   (b) Express the tangent bundle \(TK\) as the direct sum \(E_1 \oplus E_2\) of two 1-dimensional vector bundles, where \(E_2\) is the trivial bundle. Describe \(E_1\) explicitly.

5. If \(M\) is a submanifold of \(N\), show that \(TM\) is the subspace of \(TN\) consisting of vectors tangent to smooth curves in \(M\).

6. (a) Recall that \(\mathbb{R}P^n\) is the space of lines \(L\) in \(\mathbb{R}^{n+1}\) passing through the origin. The canonical line bundle over \(\mathbb{R}P^n\) is the space \(E = \{(L, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \in L\}\), or loosely speaking, “the vectors in lines through the origin”. Show that the projection \(p: E \to \mathbb{R}P^n, p(L, v) = L\), defines a vector bundle, i.e., verify the local triviality condition. Hint: think about projecting one line orthogonally onto another.

   (b) Determine whether the vector bundle \(E\) in part (a) is the trivial bundle in the special case \(n = 1\). (Give reasons for your answer, of course.)

   (c) Define the orthogonal complement \(E^\perp = \{(L, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \perp L\}\) with projection \(p: E^\perp \to \mathbb{R}P^n\) sending \((L, v)\) to \(L\). Show that this too is a vector bundle, and determine whether it is trivial when \(n = 1\).

   (d) Show that \(E \oplus E^\perp\) is isomorphic to the trivial bundle \(\mathbb{R}P^n \times \mathbb{R}^{n+1}\).
7. (a) Regarding $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$ as usual, let $E$ be the quotient space of $TS^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid v \perp x \}$ under the identifications $(x, v) \sim (-x, -v)$. Give an explanation for why $E$ is the tangent bundle $T\mathbb{R}P^n$. (This implies in particular that $E$ is a vector bundle over $\mathbb{R}P^n$.)

(b) Show that $T\mathbb{R}P^n$ is the trivial bundle for $n = 1, 3$, following the same line of reasoning as for $TS^n$ in these dimensions. (This holds also for $n = 7$ using octonions, but let’s skip that case.)

(c) One might guess that $T\mathbb{R}P^n$ is the same as the bundle $E^\perp$ in the previous problem. Show that this is false for $n = 1$. (Later we will be able to show it is false for all odd $n$.)

(d) Let $E'$ be the quotient space of the normal bundle

$$NS^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid v = tx \text{ for some } t \in \mathbb{R} \}$$

under the identifications $(x, v) \sim (-x, -v)$. Show that $E'$ is the trivial bundle $\mathbb{R}P^n \times \mathbb{R}$.

Bonus problem — it’s a little tricky:

(e*) Show that $T\mathbb{R}P^n \oplus E'$ is the direct sum of $n + 1$ copies of the canonical line bundle over $\mathbb{R}P^n$ defined in the previous problem. Hint: consider the quotient of $S^n \times \mathbb{R}^{n+1}$ under the identifications $(x, v) \sim (-x, -v)$. Show this is the direct sum of $n + 1$ copies of the bundle $(S^n \times \mathbb{R})/(x, v) \sim (-x, -v)$, and show this 1-dimensional bundle over $\mathbb{R}P^n$ is isomorphic to the canonical line bundle by thinking of $S^n \times \mathbb{R}$ as the normal bundle $NS^n$. 