1. This exercise is intended to illustrate the proof of the Theorem on page 15 of Chapter 1 in the concrete case of the continued fraction $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

(a) Write down the product $A_1 A_2 A_3 A_4 = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_4 \end{pmatrix}$ associated to $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

**Solution:** $A_1 A_2 A_3 A_4 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{5} \end{pmatrix}$

(b) Compute the four matrices $A_1$, $A_1 A_2$, $A_1 A_2 A_3$, $A_1 A_2 A_3 A_4$ and relate these to the edges of the zigzag path in the strip of triangles for $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

**Solution:** $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{pmatrix}$, $A_1 A_2 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{3} \end{pmatrix}$, $A_1 A_2 A_3 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{13}{27} \end{pmatrix}$, $A_1 A_2 A_3 A_4 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{13}{30} \end{pmatrix}$. These four matrices, together with the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, correspond to the edges of the zigzag path, with labels given by the columns of the matrices.

(c) Compute the four matrices $A_4$, $A_3 A_4$, $A_2 A_3 A_4$, $A_1 A_2 A_3 A_4$ and relate these to the successive fractions that one gets when one compute the value of $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, namely $\frac{1}{5}$, $\frac{1}{4} + \frac{1}{5}$, $\frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, and $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

**Solution:** $A_4 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{5} \end{pmatrix}$, $A_3 A_4 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{13}{27} \end{pmatrix}$, $A_2 A_3 A_4 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{313}{720} \end{pmatrix}$, $A_1 A_2 A_3 A_4 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{13}{68} \end{pmatrix}$. The second columns of these matrices are the values of the continued fractions $\frac{1}{5}$, $\frac{1}{4} + \frac{1}{5}$, $\frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, and $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

2. (a) Find all integer solutions of the equations $40x + 89y = 1$ and $40x + 89y = 5$.

**Solution:** First compute the continued fraction for $\frac{40}{89}$, which is $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$. 

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(Math 3320 Problem Set 3 Solutions)
From the strip of triangles we see that there is an edge from 9/20 to 40/89. (Note that drawing the strip is not actually necessary, as we can get 9/20 as the value of the initial segment $\frac{1}{2} + \frac{1}{4} + \frac{1}{2}$ of the continued fraction.) The determinant condition for edges implies that $9 \cdot 89 - 20 \cdot 40 = \pm 1$ and by looking at the last digit we see that in fact $9 \cdot 89 - 20 \cdot 40 = +1$. From this we get the solution $(x, y) = (-20, 9)$. This is one particular solution, and the general solution can therefore be written as $(x, y) = (-20 + 89k, 9 - 40k)$ for arbitrary integers $k$.

For the equation $40x + 89y = 5$ we multiply the solution $(-20, 9)$ of $40x + 89y = 1$ by 5 to get $(-100, 45)$. Then the general solution is $(x, y) = (-100 + 89k, 45 - 40k)$.

(b) Find another equation $ax + by = 1$ with integer coefficients $a$ and $b$ that has an integer solution in common with $40x + 89y = 1$. [Hint: use the Farey diagram.]

**Solution:** Notice that if $a/b$ is the label on any other vertex in the Farey diagram that is joined to 9/20, then (-20,9) will also be a solution of $ax + by = 1$ for the same reason that it is a solution of $40x + 89y = 1$. From the diagram we can for example choose $a/b = 4/9$. Then $(-20, 9)$ is indeed a solution of $4x + 9y = 1$. (Note that we can also read 4/9 off as the value of the initial segment $\frac{1}{2} + \frac{1}{4}$.)

3. There is a close connection between the Diophantine equation $ax + by = n$ and the congruence $ax \equiv n \mod b$, where the symbol $\equiv$ means “is congruent to”. Namely, if one has a solution $(x, y)$ to $ax + by = n$ then $ax \equiv n \mod b$, and conversely, if one has a number $x$ such that $ax \equiv n \mod b$ then this means that $ax - n$ is a multiple of $b$, say $k$ times $b$, so $ax - n = kb$ or equivalently $ax - kb = n$ so one has a solution of $ax + by = n$ with $y = -k$.

Using this viewpoint, find all integers $x$ satisfying the congruence $31x \equiv 1 \mod 71$, and then do the same for the congruence $31x \equiv 10 \mod 71$. Are the solutions unique mod 71, i.e., unique up to adding multiples of 71?

**Solution:** We want to solve $31x + 71y = 1$ so we compute the continued fraction $31/71 = \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{4}$. The initial segment $\frac{1}{2} + \frac{1}{3} + \frac{1}{2}$ equals 7/16 so there is an edge in the Farey diagram from 7/16 to 31/71 and $7 \cdot 71 - 16 \cdot 31 = 31$. This is one particular solution, and the general solution can therefore be written as $(x, y) = (31 - 71k, 7 - 16k)$ for arbitrary integers $k$. If one has a solution $(x, y)$ to $ax + by = n$, then $ax \equiv n \mod b$ if and only if $x = \frac{a}{\text{gcd}(a, b)} \mod b$, and in this case one has a solution of $ax + by = n$ with $y = \frac{n}{a} - \frac{b}{\text{gcd}(a, b)}x$.
1. Thus $31(-16) \equiv 1 \mod 71$. Multiplying by 10 gives $31(-160) \equiv 10 \mod 71$. These aren't the only possible answers since the general solution of $31x + 71y = 1$ is $(x, y) = (-16 + 71k, 7 - 31k)$ and the general solution of $31x + 71y = 10$ is $(x, y) = (-160 + 71k, 70 - 31k)$, so the general solution of $31x \equiv 1 \mod 71$ is $x = -16 + 71k$ and the general solution of $31x \equiv 10 \mod 71$ is $x = -160 + 71k$. Since the general solutions of these congruences are obtained from one particular solution by adding multiples of 71, the solutions are unique mod 71.