1. Use the Farey diagram to determine how many matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) there are having determinant 1 where the entries \( a, b, c, d \) are chosen from the numbers 1, 2, 3, 4, 5, 6, 7.

2. (a) Find a hyperbolic form \( Q(x, y) \) whose periodic separator line looks like:

\[
\begin{array}{ccccccccc}
\vdots & | & | & | & | & | & | & | \\
| & | & | & | & | & | & | \\
| & | & | & | & | & | & | \\
\vdots & | & | & | & | & | & | \\
\end{array}
\]

(b) Check your answer by computing the separator line for the form \( Q(x, y) \) that you found.

(c) Using your work for part (a), compute the value of \( \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} \).

3. Let \( Q(x, y) = ax^2 + bxy + cy^2 \) be a quadratic form of discriminant \( \Delta \). Show that the following three statements are all equivalent, i.e., each statement implies the other two:

(a) All the values \( Q(x, y) \) for primitive pairs \( (x, y) \) are odd.

(b) The coefficients \( a, b, c \) are all odd.

(c) \( \Delta \) is congruent to 5 modulo 8.

4. (a) Determine how many equivalence classes of forms of discriminant \(-356\) there are, and write down an explicit form for each equivalence class.

(b) The concept of equivalence for quadratic forms can be refined by defining two forms to be \textit{properly equivalent} if the topograph of one form can be transformed into the topograph of the other by a linear fractional transformation that preserves orientation of the Farey diagram, or in other words, a linear fractional transformation of determinant +1. Using your work in part (a), determine how many proper equiva-
lence classes there are in discriminant \(-356\), and write down an explicit form in each proper equivalence class.

5. (a) Using quadratic reciprocity, determine which primes are represented by forms of discriminant \(-24\).
(b) Do the same thing for discriminant \(+24\).
(c) Determine all the different forms of discriminant \(-24\), up to equivalence.
(d) Use the topographs of the forms in part (c) to make a reasonable conjecture as to which primes are represented by each of the forms you found. (We did not develop the tools to prove a conjecture of this sort.)

6. (a) Determine all the units in \(\mathbb{Z}[\sqrt{-14}]\), giving a reason for your answer.
(b) Determine how all the prime numbers \(p < 40\) factor into primes in \(\mathbb{Z}[\sqrt{-14}]\), and explain why your answer is correct.
(c) Show that 30 has two different factorizations into primes in \(\mathbb{Z}[\sqrt{-14}]\), where the number of prime factors is different for the two factorizations. Your answer should include an explanation of why it is correct.

7. (a) Find all equivalence classes and all proper equivalence classes (see problem 4) of quadratic forms of discriminant \(136\).
(b) Find the linear fractional transformation giving the periodicity along the separator line for the form \(x^2 - 34y^2\).
(c) Find formulas giving all integer solutions of Pell’s equation \(x^2 - 34y^2 = 1\) and also the equation \(x^2 - 34y^2 = 15\).
(d) Find the smallest unit in \(\mathbb{Z}[\sqrt{34}]\) greater than 1. Is there a smallest positive unit? Explain.

8. Let us say that a continued fraction is purely periodic if it has the special form

\[ \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \]

(a) Show that if a quadratic irrational number \(\alpha\) has a purely periodic continued fraction expansion, then \(0 < \alpha < 1\) and \(-\infty < \overline{\alpha} < -1\) where \(\overline{\alpha}\) is the conjugate of
\( \alpha \), the other root of the quadratic equation that \( \alpha \) satisfies (with integer or rational coefficients).

(b) If \( \alpha = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \) then what is the continued fraction expansion for the negative reciprocal \(-1/\alpha\) ? Explain your answer.

(c) Show that the converse of the statement in part (a) is also true. In other words, show that if \( \alpha \) is a quadratic irrational satisfying \( 0 < \alpha < 1 \) and \(-\infty < \sqrt{\alpha} < -1\), then the continued fraction expansion of \( \alpha \) is purely periodic.