13.1.1(a) Consider arbitrary sequence \((x_n)_{n \in \mathbb{N}}\) such that \(x_n \in S, \forall n \in \mathbb{N}\). There are only finitely many intervals, so at least one of them contains infinitely many terms of the sequence. Let \(I\) be this interval. We will consider the subsequence \((x_{n_k})_{k \in \mathbb{N}}\) for which the \(k\)-th term is the \(k\)-th element of \((x_n)\) that is in \(I\). As \((x_{n_k})\) is a sequence with elements in the compact interval \(I\), there is a convergent subsequence \((x_{n_{k_l}})\) that converges to \(c \in I\). This subsequence is a subsequence of \((x_n)\) and converges to an element of \(S\), so \(S\) is sequentially compact.

(b) Infinite unions of compact intervals are not always compact. For example,

\[
\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]
\]

is a union of compact intervals, but is not sequentially compact, because the sequence \(x_n = n\) has no convergent subsequences. The statement is still not true if we impose boundedness. For example, \((0, 1]\) is not sequentially compact (why?) but \([0, 1] = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1]\).

13.2.2(a) \(f(x) = \frac{1}{x} - 1\) for \(x \in (0, 1]\) and \(f(0) = 14\) is one such function. Its domain is \([0, 1]\), its range is \([0, \infty)\). The function \(f\) is not continuous.

(b) No function satisfying (a) can be continuous. The domain is a compact interval, so all continuous functions with this domain are bounded. As this function is not bounded, it cannot hope to be continuous.

13.3.2 Write the polynomial as \(p(x) = a_n x^n + \ldots + a_0\). We will show that \(p(x)\) has a minimum if \(a_n > 0\). If \(a_n > 0\) from a previous homework we know that \(\lim_{x \to \pm \infty} p(x) = \infty\). In particular, this implies that there is \(M > 0\) such that for \(x < -M\) and \(x > M\), \(p(x) \geq a_0\). As \([-M, M]\) is a compact interval and \(p(x)\) is continuous, \(p(x)\) has a minimum on \([-M, M]\), call this minimum \(m\). For every \(x \in \mathbb{R}\), if \(x \notin [-M, M]\) then \(x > a_0 = p(0) \geq m\), and if \(x \in [-M, M]\) then \(x \geq m\), so \(p\) has a minimum at \(m\). The same sort of argument shows that if \(a_n < 0\) that \(p(x)\) has a maximum on \((-\infty, \infty)\).

13.4.1 (a) Let \(I\) be a compact interval. As its image is a compact interval, there is \(a, b\) such that \(f(I) = [a, b]\). By the definition of image, as \(a\) and \(b\) are in the image of \(f\), there is \(x \in I\) such that \(f(x) = a\) and \(y \in I\) such that \(f(y) = b\). Therefore \(f\) has a maximum at \(y\) and a minimum at \(x\).

(b) Notice that the image is an interval, so if \(I = [\alpha, \beta]\), then \(f(\alpha), f(\beta)\) is in the image, and so is every element in between them. Hence for every \(c\) between \(f(\alpha)\) and \(f(\beta)\), there is \(z \in I\) such that \(f(z) = c\), and thus \(f\) has the IVT.

13.4.2 Consider the function \(f(x) = \sin(\frac{1}{x}), x \neq 0\) and \(f(0) = 0\). This is not a continuous function, as there is a sequence \((x_n)\) converging to 0 such that \(f(x_n) = 1\) for all \(n\). It does satisfy the property of the previous question.

If \([a, b]\) is a compact interval, either it does not contain 0, and \(f\) is a continuous function on \([a, b]\), and hence has image a compact interval, or it contains 0, and then the image of \(f\) on \([a, b]\) is \([0, 1]\).

13.5.2 Let \(p\) be a period of \(f\), periodic function defined on \(\mathbb{R}\). As \(f\) is continuous, by uniform continuity on compact intervals, \(f\) is uniformly continuous on \([-p, p]\). This implies that given \(\epsilon > 0\), there exists \(\delta > 0\), which we may assume is also smaller than \(p/2\), such that if \(x, y \in [-p, p]\) and \(|x - y| < \delta\) then \(|f(x) - f(y)| < \epsilon\). We will show that in fact, for any \(x, y \in \mathbb{R}\), if \(|x - y| < \delta\) then \(|f(x) - f(y)| < \epsilon\). If \(x, y \in \mathbb{R}\) satisfy that \(|x - y| < p/2\) then either \(np < x, y < (n + 1)p\) for some unique \(n\), or \(x < mp < y\) for some unique \(m\). In the first case, \(|x - y| = |(x - np) - (y - np)| < \delta\), and \(x - np, y - np \in [0, p]\). Then \(|f(x) - f(y)| = |f(x - np) - f(y - np)| < \epsilon\). In the second case \(x - mp \in [-p, 0]\) and \(y - mp \in [0, p]\). As \(|x - y| = |(x - mp) - (y - mp)| < \delta\), then \(|f(x) - f(y)| = |f(x - mp) - f(y - mp)| < \epsilon|.

13.5.3 Given \(\epsilon > 0\) we want to find \(\delta > 0\) such that if \(x, y \in I \cup J\) and \(|x - y| < \delta\) then \(|f(x) - f(y)| < 2\epsilon\). As \(f\) is uniformly continuous on \(I\), there is a \(\delta_1\) such that for \(|x - y| < \delta_1\), \(|f(x) - f(y)| < \epsilon\) as long as \(x, y \in I\). Similarly, there is a \(\delta_2\) such that this condition holds as long as \(y, x \in J\). Let \(\delta = \min(\delta_1, \delta_2)\). Certainly this \(\delta\) still works for \(I\) and \(J\). Now suppose that \(x \in I\) and \(x \notin J\), \(y \in J\) and \(y \notin I\), and \(|x - y| < \delta\). WLOG, \(x < y\). As \(I \cap J\) and \(I\) and \(J\) are intervals, there is \(z \in I \cap J\) and \(x < z < y\). Therefore \(|x - z| < \delta\), \(y < z < y\), \(|z - y| < \delta\) and \(y, z \in J\), so we can apply uniform continuity separately. Hence \(|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq 2\epsilon|.

13.5.4 There are several ways to show that \(f(x) = \sqrt{x}\) is uniformly continuous on \([0, \infty)\). One way is to apply the last problem and show that it is uniformly continuous on \([0, 1]\) and \([1, \infty)\). It is uniformly continuous on \([0, 1]\) because it is a continuous function on a compact interval. Now we prove from the definition that it is uniformly
continuous on $[1, \infty)$. Given $\epsilon > 0$, for $\delta = \epsilon$, if $|x - y| \leq \delta$ then

$$|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq |x - y| < \epsilon$$

The first equality comes from thinking of $x - y$ as a difference of squares. The second comes from the fact that $x, y \geq 1$. Therefore $\sqrt{x}$ is uniformly continuous on $[1, \infty)$, and the desired result follows.

13.5.5 As $\ln(x)$ is continuous at 1, given $\epsilon > 0$, there is $\delta > 0$ such that if $|x - 1| < \delta$, then $|\ln(x) - \ln(1)| < \epsilon$. For $x, y \in \mathbb{R}$, such that $|x - y| < \delta$ the $|x/y - 1| < \delta/x \leq \delta$ (as $x \geq 1$). Therefore $|\ln(x) - \ln(y)| = |\ln(x/y)| = |\ln(x/y) - \ln(1)| < \epsilon$. Hence $\ln(x)$ is uniformly continuous on $[1, \infty)$.

13.5.6 (a) For $\epsilon > 0$, let $\delta = \epsilon/K$. Then if $|x - y| < \delta$,

$$|f(y) - f(x)| \leq K|y - x| < K\epsilon/K = \epsilon$$

so $f$ is uniformly continuous.

(b) $\sqrt{x}$ is uniformly continuous, but does not satisfy this condition (called Lipschitz continuity). We will show that for every $K$ there is $x, y$ pair such that the slope of the secant between $x$ and $y$ is greater than $K$: Let $x = 0, y = \frac{1}{4K^2}$

$$\frac{|\sqrt{x} - \sqrt{y}|}{|x - y|} = \frac{\sqrt{y}}{y} = \frac{\sqrt{1}}{\sqrt{y}} = 2K > K$$