Taylor series Homework set solutions.

17.1.1 Let \( f(x) = (1 + x)^r \). Then, by the usual differentiation formulas, we have that

\[
f^{(k)}(x) = r(r - 1) \cdots (r - k + 1)(1 + x)^{r-k}.
\]

Therefore,

\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{r(r - 1) \cdots (r - k + 1)}{k!} x^k.
\]

Observation 1 The formula

\[
\binom{r}{k} := \frac{r(r - 1) \cdots (r - k + 1)}{k!}
\]

is sometimes used to represent the coefficient appearing on the right hand side of equation (1). Observe that when \( r \geq k \) is an integer, this formula agrees with its usual definition.

17.2.1 We will prove Lemma 17.2 using induction.

Base Case. By the usual Rolle’s theorem, if \( f(a) = f(b) = 0 \), then there exists \( c \in (a, b) \) such that \( f'(c) = f^{(0+1)}(c) = 0 \).

Inductive Step. Now assume that the result holds up to a fixed \( n \in \mathbb{N} \), that is, assume that if \( f^{(n+1)} \) exists on \( [a, b] \) and \( f(a) = f'(a) = \ldots = f^{(n)}(a) = f(b) = 0 \), then there exists \( c \) between \( a \) and \( b \) such that \( f^{(n+1)}(c) = 0 \).

We want to show that the result holds for \( n + 1 \), that is, we want to show that if \( f^{(n+2)} \) exists on \( [a, b] \) and \( f(a) = f'(a) = \ldots = f^{(n+1)}(a) = f(b) = 0 \), then there exists \( c \) between \( a \) and \( b \) such that \( f^{(n+2)}(c) = 0 \). Let \( g(x) = f'(x) \). By the usual Rolle’s theorem, there exists \( c_0 \in (a, b) \) such that

\[
g(c_0) = f'(c_0) = 0.
\]

Hence \( g^{(n+1)} = f^{(n+2)} \) exists and \( g(a) = g'(a) = \ldots = g^{(n)}(a) = g(c_0) = 0 \). Therefore, by induction hypothesis, there exists \( c \) between \( a \) and \( c_0 \) such that \( g^{(n+1)}(c) = 0 \). Since \( (a, c_0) \subset (a, b) \), we conclude that

\[
f^{(n+2)}(c) = g^{(n+1)}(c) = 0 \quad \text{for some } c \in (a, b),
\]

as we wanted to show.
17.2.2 a) Assume that

\[ P(x) = \sum_{k=0}^{n} b_k x^k = b_0 + b_1 x + \cdots + b_n x^n \]

is a polynomial of degree \( n \). If \( x = u + a \), then we can rewrite the above expression as

\[
\begin{align*}
P(x) &= P(u + a) \\
&= \sum_{k=0}^{n} b_k (u + a)^k \\
&= \sum_{k=0}^{n} b_k \left( \sum_{l=0}^{k} \binom{k}{l} a^{k-l} u^l \right) \\
&= \sum_{k=0}^{n} \sum_{l=0}^{k} b_k \binom{k}{l} a^{k-l} u^l \\
&= \sum_{l=0}^{n} \left[ \sum_{k=l}^{n} b_k \binom{k}{l} a^{k-l} \right] u^l, \\
\end{align*}
\]

where the last equality follows from the identities of iterated sums. Substituting \( u = x - a \) in (2) we get

\[
P(x) = \sum_{l=0}^{n} \left[ \sum_{k=l}^{n} b_k \binom{k}{l} a^{k-l} \right] (x - a)^l.
\]

On the other hand, using the usual formulas for the derivatives, we obtain that

\[
P^{(l)}(x) = \sum_{k=l}^{n} b_k (k-1) \cdots (k-l+1) x^{k-l}.
\]

Hence

\[
T_n(x) = \sum_{l=0}^{n} \frac{P^{(l)}(a)}{l!} (x - a)^l = \sum_{l=0}^{n} \frac{\sum_{k=l}^{n} b_k (k-1) \cdots (k-l+1) a^{k-l}}{l!} (x - a)^l.
\]

Comparing equations (3) and (4), we conclude that

\[ P(x) = T_n(x). \]

b) Observe that, if we set \( x = u - 1 \),

\[
P(x) = (u - 1)^3 - 2(u - 1) + 2 = u^3 - 3u^2 + u + 3 = (x + 1)^3 - 3(x + 1)^2 + (x + 1) + 3.
\]

On the other hand, we have that

\[
P(x) = x^3 - 2x + 2, \quad \text{hence} \quad P(-1) = 3.
\]
\[
P'(x) = 3x^2 - 2, \quad \text{hence} \quad P'(-1) = 1.
\]
\[
P''(x) = 6x, \quad \text{hence} \quad P''(-1) = -6.
\]
\[
P'''(x) = 6, \quad \text{hence} \quad P'''(-1) = 6.
\]
Therefore, \[ P(x) = 3 + (x + 1) + \frac{(-6)}{2!}(x + 1)^2 + \frac{6}{3!}(x + 1)^3 = 3 + (x + 1) - 3(x + 1)^2 + (x + 1)^3. \]

17.2.3 We will prove this result using induction on \( n \).

**Base Case.** For \( n = 0 \), the problem is asking us to show that if \( g(a) = g(b) = 0 \), then there exist \( c \in (a, b) \) such that \( g^{(0+1)}(c) = g'(c) = 0 \). But this is precisely the statement of Rolle’s theorem.

**Inductive Step.** Now assume that the result holds up to a fixed \( n \in \mathbb{N} \). If we set \( a_0 = a \), and \( a_{n+1} = b \), then this is equivalent to assuming that if \( g \) has \((n + 1)\) derivatives, and \[ g(a_0) = g(a_1) = \cdots g(a_{n+1}) = 0, \]
then there exists \( c \in (a, b) \) such that \( g^{(n+1)}(c) = 0 \).

We want to prove that this result holds for \( n+1 \), that is, we want to show that if \( g \) has \((n + 2)\) derivatives, and if \[ g(a_0) = g(a_1) = \cdots g(a_{n+2}) = 0, \]
then there exists \( c \in (a, b) \) such that \( g^{n+2}(c) = 0 \). Let’s set \( f(t) = g'(t) \). Then, by Rolle’s theorem, there exists \( c_k \in (a_k, a_{k+1}) \) for \( k = 0, 1, \ldots, n+1 \) such that \[ f(c_k) = g'(c_k) = 0 \quad \text{for} \quad k = 0, \ldots, n+1. \]

Hence, by induction hypothesis, there exists \( c \in (c_0, c_{n+1}) \subset (a, b) \) such that \[ f^{(n+1)}(c) = g^{(n+2)}(c) = 0, \]
as we wanted to show.

17.2.4 a) Since \( f(a) = f(b) = 0 \), then, by Rolle’s theorem, there exists \( c_0 \in (a, b) \) such that \( f'(c_0) = 0 \). Applying Rolle’s theorem again on the intervals \([a, c_0]\) and \([c_0, b]\) for the function \( g(x) = f'(x) \), we get that there exists \( c_1 \in [a, c_0] \) and \( c_2 \in [c_0, b] \) such that \[ f''(c_1) = f''(c_2) = 0. \]

Yet another application of Rolle’s theorem gives us a \( c \in (c_1, c_2) \) such that \( f'''(c) = 0 \).

b) Observe that \[ f'(x) = 2(x - a)(x - b)^2 + 2(x - a)^2(x - b). \]

Hence \( f(a) = f(b) = f'(a) = f'(b) = 0 \), that is, the hypothesis of part a) apply. Now, by the usual differentiation formulas, we obtain that \[ f'''(x) = 24x - 12a - 12b. \]
Hence, the equation \( f'''(c) = 0 \) is equivalent to the equation

\[
24c - 12a - 12b = 0 \\
c = \frac{a + b}{2}
\]

17.3.1 Let \( f(x) = e^{-x} \). Then,

\[
T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} x^k = 1 - x + \frac{x^2}{2}.
\]

We want to estimate the magnitude of the residue function

\[ R_2(x) = f(x) - T_2(x) \quad \text{for } x \in [0, 0.1]. \]

By Taylor’s theorem with Lagrange remainder, we know that for all such \( x \), there exists a \( c \) between 0 and \( x \) such that

\[ R_2(x) = \frac{f^{(3)}(c)}{3!} x^3. \]

So the magnitude of the error is given by

\[
|R_2(x)| = \left| e^{-c} \right| \left| \frac{x^3}{3!} \right| \quad \text{for some } 0 \leq c \leq x \leq 0.1.
\]

Therefore, we get the bound

\[
|R_2(x)| \leq e^0 \left( \frac{0.1}{3!} \right)^3 = \left( \frac{0.1}{6} \right)^3.
\]

17.3.2 Let \( f(x) = \cos x \). Observe that, since \( f'''(0) = -\sin 0 = 0 \), the expression \( 1 - x^2/2 \) is not only a degree 2 approximation, but actually a degree 3 approximation! (In other words, \( T_2(x) = T_3(x) = 1 - x^2/2 \).) Just as in the last problem, we obtain an approximation of the remainder function of the form

\[
|R_3(x)| = \left| \frac{f^{(4)}(c)}{4!} x^4 \right| = \left| \cos c \right| \frac{|x|^4}{24} \quad \text{for some } 0 \leq c \leq x.
\]

Therefore, if we want to bound the error by .0001 it suffices to choose an interval of the form \([ -b, b] \) with

\[
\frac{b^4}{24} \leq .0001 \\
b \leq \sqrt[4]{24/10} \approx 0.22136
\]

17.3.3 Let \( f(x) = \sin x \). We want to bound the remainder function \( T_n(x) \) by .0001 for \( |x| < .5 \). Proceeding as in the last problem, we can quickly get a bound of the form

\[
|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| < \frac{1}{2^{n+1}(n+1)!}.
\]

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Now, observe that $3840 = 2^{14+1}(4 + 1)! < 10, 000 < 2^{5+1}(5 + 1)! = 46,080$. Hence, it is enough to take $T_5$. However, since $T_4 = T_5$ (for similar reasons as in problem 17.3.2) it suffices to take $n = 4$.

17.3.4 Let $f(x) = \cos x$. Once again, we observe that since $T_4 = T_5$, we have an estimate of the form

$$|R_4(x)| = |R_5(x)| \leq \frac{\pi^6}{6!}.$$ 

Now, since

$$(.1)^6/6! < (.1)^8,$$

we conclude that $T_4(.1)$ is a good enough approximation to $\cos .1$. Calculating we get the approximation

$$\cos .1 \approx 1 - (.1)^2/2 + (.1)^4/24 \approx .9950041.$$

17-1 a) Let $P(x)$ be a polynomial of degree $n$. Then,

$$P(x) = \sum_{l=0}^{n} c_l (x-a)^l,$$

where $c_l = P^{(l)}(a)/l!$. Therefore, if we assume that

$$P(a) = P'(a) = \ldots = P^{(k-1)}(a) = 0, \quad P^{(k)}(a) \neq 0,$$

then

$$P(x) = (x-a)^k \left( \sum_{l=k}^{n} c_l (x-a)^{l-k} \right).$$

If we set

$$Q(x) = \sum_{l=k}^{n} c_l (x-a)^{l-k},$$

then it is clear that $P(x) = (x-a)^k Q(x)$ and $Q(a) = c_k = P^{(k)}(a)/k! \neq 0$, that is, $a$ is $k$-fold zero of $P(x)$.

Now assume that $a$ is a $k$-fold zero of $P(x)$, that is, assume that there exists a polynomial $Q(x)$ such that $P(x) = (x-a)^k Q(x)$ and $Q(a) \neq 0$. Expressing the polynomial $Q(x)$ as

$$Q(x) = \sum_{l=0}^{m} b_l (x-a)^l,$$

we obtain that

$$P(x) = (x-a)^k \sum_{l=0}^{m} b_l (x-a)^l$$

$$= \sum_{l=0}^{m} b_l (x-a)^{k+l}$$

$$= \sum_{l=k}^{k+m} b_{l-k} (x-a)^l$$
with $b_0 \neq 0$. From this expression, it is immediate that
\[ P(a) = P'(a) = \ldots = P^{(k-1)}(a) = 0, \quad \text{and} \quad P^{(k)}(a) = b_0k! \neq 0, \]

b) Assume that $a$ is a double zero of the polynomial $P(x) = 2x^3 - bx^2 + 1$. Then, according to part a) we should have that
\[ 0 = P'(a) = 6a^2 - 2ba = a(6a - 2b). \]
Hence, the only options for $a$ are $a = 0$ or $a = b/3$. Since $P(0) \neq 0$, we find ourselves looking for a value of $b$ such that
\[
\begin{align*}
0 &= f(a) = f(b/3) = 2\left(\frac{b}{3}\right)^3 - b\left(\frac{b}{3}\right)^2 + 1 \\
-1 &= \frac{2}{3}\left(\frac{b^3}{9}\right) - \left(\frac{b^3}{9}\right) \\
-1 &= -\frac{1}{3}\left(\frac{b^3}{9}\right) \\
b^3 &= 27 \\
b &= 3.
\end{align*}
\]
Plugging back into our equations, we can see that, effectively, for $b = 3$ the value $a = 1$ is a double zero of the polynomial $P(x)$.

c) If in exercise 17.2.3 we take all the roots to be equal to $a$ instead of being all different, then part a) of this problem says that the conclusion in exercise 17.2.3 still holds and is equivalent to the Extended Rolle's Theorem.