Problem 1: 7.2.1

Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

Solution: Let

\[
    a_n = \begin{cases} 
        \frac{1}{n^2} & \text{n even} \\
        0 & \text{n odd} 
    \end{cases} \quad b_n = \begin{cases} 
        0 & \text{n even} \\
        \frac{1}{n^2} & \text{n odd} 
    \end{cases}.
\]

Observe $\sum_{i=1}^{k} a_n \leq \sum_{i=1}^{k} \frac{1}{n^2} \leq \frac{\pi^2}{6}$, so the partial sums of $\sum_{n=1}^{\infty} a_{2n}$ are increasing and bounded above, so $\sum_{n=1}^{\infty} a_n$ converges. By a similar argument, $\sum_{n=1}^{\infty} b_n$ converges.

Since $a_n + b_n = \frac{1}{n^2}$, $\sum a_n$ converges and $\sum b_n$ converges, by linearity:

\[
    \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n
\]

Moreover:

\[
    \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}
\]

Hence:

\[
    \frac{\pi^2}{6} = \frac{\pi^2}{24} + \sum_{n=1}^{\infty} b_n \Rightarrow \sum_{n=1}^{\infty} b_n = \frac{\pi^2}{8}
\]

and the $2n + 1$ partial sum of $\sum_{n=1}^{\infty} b_n$ is the $n$th partial sum of $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$, so the respective sequences of partial sums converge to the same limit by the subsequence theorem.

Problem 2: 7.2.3

Let $a_n, b_n$ be non-negative series and let $\sum a_n$ and $\sum b_n$ converge. Prove $\sum a_n b_n$ converges using the methods:

i. Use an inequality relating the partial sums of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} a_n b_n$.

Proof. Let $s_k, s'_k, s''_k$ denote the $k$th partial sums of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} a_n b_n$ respectively.

We know that $s_k, s'_k$ converge to some values $s, s'$ respectively, so by the product theorem for sequences, the sequence $s_k s'_k \to s s'$. Furthermore, $a_n, b_n$ are non-negative, so $s_k, s'_k$ are increasing sequences. Consequently, $s_k \leq s, s'_k \leq s'$, so $s_k s'_k < s s'$ for all $k$. Since $a_i b_j \geq 0$ for all $i, j$:

\[
    s''_k = \sum_{n=1}^{k} a_n b_n \leq \left( \sum_{n=1}^{k} a_n \right) \left( \sum_{m=1}^{k} b_m \right) \leq s_k s'_k,
\]

so $s''_k$ is an increasing (because each $a_n b_n \geq 0$) sequence which is bounded above. Hence $\sum_{n=1}^{\infty} a_n b_n = \lim_{k \to \infty} s''_k$ converges by the completeness axiom. \qed
ii. By using the comparison test.

Proof. Since $a_n$ is non-negative, $a_n = |a_n|$ and $a_n$ is absolutely convergent. Since $\sum b_n$ converges, $b_n \to 0$ as $n \to \infty$ by the divergence test and in particular, $b_n$ is bounded.\footnote{Given $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|b_n| < \epsilon$, then $M = \max(b_1, \ldots, b_N, 1)$ is an upper bound for $b_n$.} The result now follows by problem 7.3.2 (see below). \hfill \blacksquare

Problem 3: 7.2.5
Let $\sum_{n=0}^{\infty} a_n$ be a convergent series with the sum $S$. Create a sequence $b_k$ such that $b_k = 2^n + a_{2^n + 1}$.

Proof. Observe the $2^n + 1$th partial sum of $\sum_{n=0}^{\infty} a_n$ is the $k$th partial sum of $\sum_{n=0}^{\infty} b_n$, so the partial sums of the first series form a subsequence of the partial sums of the second. The result now follows by the subsequence theorem. \hfill \blacksquare

Problem 4: 7.3.5
Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent and let $b_n$ be a bounded sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof. Let $|b_n| \leq M$ for some $M \in \mathbb{R}$. Then $|a_n b_n| \leq M|b_n|$, so by the comparison test for series: $\sum |a_n b_n|$ converges. By the absolute convergence theorem, $\sum a_n b_n$ converges. \hfill \blacksquare

Problem 5: 7.3.6
Let $\sum a_n$ be a conditionally convergent series. Prove there exist infinitely many positive and negative terms of $a_n$.

Proof. Suppose toward a contradiction there are only finitely many negative terms of $a_n$. Then there exists $N \in \mathbb{N}$ such that for $n \geq N$, $a_n \geq 0$. We know $\sum_{n=N}^{\infty} a_n$ converges by the tail convergence theorem and:

$$\sum_{n=N}^{\infty} |a_n| = \sum_{n=N}^{\infty} a_n$$

so by the tail convergence theorem $\sum_{n=1}^{\infty} |a_n|$ converges. Thus $\sum a_n$ is absolutely convergent, contradicting our hypothesis.

Observe that $\sum -a_n$ is also conditionally convergent, so the above argument shows that $-a_n$ has infinitely many positive terms. Hence $a_n$ has infinitely many negative terms. \hfill \blacksquare