1.2.12: **Problem:** Let \( \mathbf{v} = (2, 3) \). Suppose \( \mathbf{w} \in \mathbb{R}^2 \) is perpendicular to \( \mathbf{v} \), and that \( \|\mathbf{w}\| = 5 \). This determines \( \mathbf{w} \) up to sign. Find one such \( \mathbf{w} \).

**Solution:** If \( \mathbf{w} = (a, b) \), by orthogonality to \( \mathbf{v} \), we get, \( \mathbf{v} \cdot \mathbf{w} = 2a + 3b = 0 \). This means \( a = -\frac{3b}{2} \). So \( \mathbf{w} = (-\frac{3b}{2}, b) \) for some value of \( b \).

Using the fact that \( \|\mathbf{w}\| = 5 \), we can figure out \( b \):

\[
5 = \|\mathbf{w}\| = \sqrt{\frac{9b^2}{4} + b^2} = \sqrt{\frac{13b^2}{4}}.
\]

Squaring, we get, \( b^2 = \frac{100}{13} \). So \( b = \pm \frac{10}{\sqrt{13}} \). Thus the two possible answers are, \( \mathbf{w} = \pm (-\frac{15}{\sqrt{13}}, \frac{10}{\sqrt{13}}) \).

1.2.18: **Problem:** Find all values of \( x \) such that \( (x, 1, x) \) and \( (x, -6, 1) \) are orthogonal.

**Solution:** To be orthogonal, their dot product must be 0.

That is, \( (x, 1, x) \cdot (x, -6, 1) = x^2 - 6 + x = 0 \).

This quadratic equation factors as: \( (x + 3)(x - 2) = 0 \), so the only two values are \( x = -3, 2 \), for which the two vectors will be orthogonal.

1.3.16: **Problem:** Find an equation for the plane that passes through:

(a) \( (0, 0, 0), (2, 0, -1), \) and \( (0, 4, -3) \).

(b) \( (1, 2, 0), (0, 1, -2), \) and \( (4, 0, 1) \).

(c) \( (2, -1, 3), (0, 0, 5), \) and \( (5, 7, -1) \).

**Solution**

(a): Let \( \mathbf{P} = (0, 0, 0), \mathbf{Q} = (2, 0, -1), \) and \( \mathbf{R} = (0, 4, -3) \)

Then the vectors \( \overrightarrow{PQ} = (2, 0, -1) \), and \( \overrightarrow{PR} = (0, 4, -3) \) are on the plane. A normal vector to this plane is \( \mathbf{n} = (2, 0, -1) \times (0, 4, -3) \), which we calculate...
as follows:

\[
\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 4 & -3 \end{vmatrix} = (0 - (-4))i - (-6 - 0)j + (8 - 0)k = 4i + 6j + 8k.
\]

So the equation of the plane must be of the form, \(4x + 6y + 8z + D = 0\). And since \(P = (0, 0, 0)\) is on the plane, it must satisfy this equation, so plugging in \((x, y, z) = (0, 0, 0)\), we get \(D = 0\).

Thus an equation of the plane passing through \((0, 0, 0), (2, 0, -1),\) and \((0, 4, -3)\) is \(4x + 6y + 8z = 0\).

\((b)\): Let \(P = (1, 2, 0), Q = (0, 1, -2),\) and \(R = (4, 0, 1)\)

Then the vectors \(\overrightarrow{PQ} = (-1, -1, -2),\) and \(\overrightarrow{PR} = (3, -2, 1)\) are on the plane. A normal vector,

\[
\vec{n} = \begin{vmatrix} i & j & k \\ -1 & -1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (-1 - (4))i - (-1 - (-6))j + (2 - (-3))k = -5i - 5j + 5k.
\]

So the equation of the plane must be of the form, \(-5x - 5y + 5z + D = 0\). And since \(P = (1, 2, 0)\) is on the plane, it must satisfy this equation, so plugging in \((x, y, z) = (1, 2, 0)\), we get \(-5(1) - 5(2) + D = 0\). So \(D = 15\).

Thus an equation of the plane passing through \((1, 2, 0), (0, 1, -2),\) and \((4, 0, 1)\) is \(-5x - 5y + 5z + 15 = 0, \) or \(x + y - z = 3\).

\((c)\): Let \(P = (2, -1, 3), Q = (0, 0, 5),\) and \(R = (5, 7, -1)\)

Then the vectors \(\overrightarrow{PQ} = (-2, 1, 2),\) and \(\overrightarrow{PR} = (3, 8, -4)\) are on the plane.

\[
\vec{n} = \begin{vmatrix} i & j & k \\ -2 & 1 & 2 \\ 3 & 8 & -4 \end{vmatrix} = (-4 - (16))i - (8 - 6)j + (-16 - 3)k = -20i - 2j - 19k.
\]
So the equation of the plane must be of the form, \(-20x - 2y - 19z + D = 0\).
And since \(Q = (0, 0, 5)\) is on the plane, \(-19(5) + D = 0\). So \(D = 95\).

Thus an equation of the plane passing through \((2, -1, 3), (0, 0, 5)\), and \((5, 7, -1)\) is \(-20x - 2y - 19z + 95 = 0\), or \(20x + 2y + 19z = 95\).

1.3.22: **Problem:** Find the intersection of the two planes with equations
\(3(x - 1) + 2y + (z + 1) = 0\) and
\((x - 1) + 4y - (z + 1) = 0\)

**Solution:** The coefficients of \(x, y, z\) do not form parallel vectors, so the two planes must not be parallel. Thus they must intersect in a line. We will find a parametric form of their line of intersection. We have,

\(4(x - 1) + 6y = 0\) – by adding the two equations, (to eliminate \(z\)), and
\(-10y + 4(z + 1) = 0\) – by subtracting 3 times the second from the first, (to eliminate \(x\)).

So
\[
\begin{align*}
\frac{1}{4} &\quad y = -1 + \frac{10y}{4} \\
\frac{1}{4} &\quad x = 1 - \frac{6y}{4}
\end{align*}
\]

Now, by setting \(y = t\) (a parameter), we obtain that the line of intersection must be:

\[
\begin{pmatrix}
x \\ y \\ z
\end{pmatrix} = \begin{pmatrix}
1 - \frac{6t}{4} \\ t \\ -1 + \frac{10t}{4}
\end{pmatrix}
\]

This could also be written as,

\[
\begin{pmatrix}
1 & -3/2 \\ 0 & 1 \\ -1 & 5/2
\end{pmatrix} + t \begin{pmatrix}
1 & -3s \\ 2s & 5s - 1
\end{pmatrix}, \text{ for } s = t/2.
\]
1.4.4: **Problem:** Describe the geometric meaning of the following mappings in cylindrical coordinates:

(a) \((r, \theta, z) \mapsto (r, \theta, -z)\).
(b) \((r, \theta, z) \mapsto (r, \theta + \pi, -z)\).
(c) \((r, \theta, z) \mapsto (-r, \theta - \pi/4, z)\).

**Solution:**

(a): Since \(r, \theta\) remain unchanged, only the height along the \(z\)-axis of a point is changed. Each point is reflected across the \(xy\)-plane (or \(r\theta\)-plane), as if there were a mirror on this plane. Points on the \(xy\)-plane remain fixed.

(b): Adding \(\pi\) to \(\theta\) rotates each point about the \(z\)-axis by 180 degrees. This changes the \((x, y)\) values to \((-x, -y)\).

Simultaneously, since \(z\) is changed to \(-z\), the resulting mapping takes \((x, y, z)\) to \(-(x, y, z)\). Thus this mapping amounts to "reflecting across the origin", or simply negating every vector.

(c): If \(r\) is not allowed to be negative, this mapping does not make sense, and points were not removed for stating this.

However, if \(r\) is allowed to be negative to describe mappings, then \(r \mapsto -r\) has the same affect as \(\theta \mapsto \theta + \pi\), or of rotating about the \(z\)-axis by 180 degrees. \(\theta \mapsto \theta - \pi/4\) has the affect of rotating about the \(z\)-axis clockwise (looking from above) by 45 degrees. All in all this mapping rotates about the \(z\)-axis counterclockwise by \(180 - 45 = 135\) degrees.

1.4.10: **Problem:** Describe the following solids using inequalities. State the coordinate system used.

(a) A cylindrical shell 8 units long, with inside diameter 2 units and outside diameter 3 units

(b) A spherical shell with inside radius 4 units and outside radius 6 units

(c) A hemisphere of diameter 5 units
(d) A cube of side length 2

**Solution:**

(a): This is best described in cylindrical coordinates. Note for diameter $= 2$ use radius $= 1$.

$$1 \leq r \leq 3/2 \text{ and } |z| \leq 4 \text{ (one possibility)}$$

(b): This is best described in spherical coordinates.

$$4 \leq \rho \leq 6$$

(c): This is also best described in spherical coordinates.

$$\rho \leq 5/2 \text{ and } 0 \leq \theta \leq \pi$$

or

$$\rho \leq 5/2 \text{ and } 0 \leq \phi \leq \pi/2$$

Note, $0 \leq \phi \leq \pi$ would give the whole sphere. If the sphere is assumed hollow, use $\rho = 5/2$ instead of $\rho \leq 5/2$.

(d): This is best described in Cartesian (or rectangular) coordinates as:

$$|x| \leq 1$$

$$|y| \leq 1$$

$$|z| \leq 1$$