The toric $h$-vector

Originally introduced to correspond to the Betti numbers of the intersection cohomology of toric varieties associated to rational polytopes, the toric $h$-vector can be defined for any finite graded poset with a minimum element $\hat{0}$ and a maximum element $\hat{1}$. With the exceptions noted below, we follow Stanley’s presentation in EC 1.

Let $P$ be a finite graded poset with $\hat{0}$ and $\hat{1}$ and let $\rho$ be the rank function of $P$. Define two polynomials $\hat{h}$ and $\hat{g}$ inductively as follows.

- $\hat{h}(1, t) = \hat{g}(1, t) = 1$. Here $1$ is the poset with only one element $\hat{1} = \hat{0}$.
- If the rank of $P$ is $d+1$, then $\hat{h}(P, t)$ has degree $d$. Write $\hat{h}(P, t) = \hat{h}_d + \hat{h}_{d-1}t + \hat{h}_{d-2}t^2 + \cdots + \hat{h}_0t^d$. Then define $\hat{g}(P, t) = \hat{h}_d + (\hat{h}_{d-1} - \hat{h}_d)t + (\hat{h}_{d-2} - \hat{h}_{d-1})t^2 + \cdots + (\hat{h}_{d-m} - \hat{h}_{d-m+1})t^m$, where $m = \lfloor d/2 \rfloor$.

[NOTE: Our $\hat{h}_i$ is $\hat{h}_{d-i}$ in EC 1.]
- If the rank of $P$ is $d+1$, then define

$$\hat{h}(P, t) = \sum_{x \in P, x \neq \hat{1}} \hat{g}([\hat{0}, x], t)(t - 1)^{d - \rho(x)}.$$

From here on we will write $\hat{h}([\hat{0}, x], t)$ as $\hat{h}(x, t)$. Similarly for $\hat{g}(x, t)$.

Induction shows that if $B_d$ is the face poset of the $(d-1)$-simplex, then $\hat{h}(B_d, t) = 1 + t + \cdots + t^{d-1}$ and $\hat{g}(B_d, t) = 1$. From this it follows that if $P$ is the face poset of a simplicial complex $\Delta$ with $\hat{1}$ adjoined, then $\hat{h}_i(P, t) = h_i(\Delta)$.

**Example 0.1.** Figure 2 shows the Hasse digram, $P$, of the face poset of the cell decomposition of the torus depicted in figure 1. The rank one and rank two elements correspond to simplices. The four rank two elements, which correspond to the four rectangles of the cell decomposition have $\hat{g} = 1 + t$. So,

$$\hat{h}_P(t) = (t - 1)^3 + 4(t - 1)^2 + 8(t - 1) + 4(t + 1) = t^3 + t^2 + 7t - 1.$$ 

Hence $\hat{h}_0 = 1, \hat{h}_1 = 1, \hat{h}_2 = 7$ and $\hat{h}_3 = -1$.

**Theorem 0.2.** Let $P$ be a semi-Eulerian poset of rank $d + 1$ and let $\Delta_P$ be the reduced order complex of $P$. Then

$$\hat{h}_{d-i} - \hat{h}_i = (-1)^i \binom{d}{i} [\chi(\Delta_P) - \chi(S^{d-1})].$$
Remark 0.3. If $\Delta$ is a $(d - 1)$-dimensional pure simplicial complex, then $F(\Delta) \cup \hat{1}$ is semi-Eulerian if and only if $\chi(\text{lk}_\Delta \sigma) = \chi(S^{d-1-|\sigma|})$ for all nonempty faces $\sigma$ of $\Delta$. This includes triangulations of compact manifolds. The $f$-vector form of the above theorem for face posets of simplicial complexes is originally due to Klee [A combinatorial analog of Poincaré’s duality theorem, Can. J. Math., 16 (1964), 517–531].

Proof. The proof is a variation of Stanley’s proof of this equation in the special case that $P$ is Eulerian in EC 1. Let $u = t - 1$. Multiply (1) by $u$ and add $\hat{g}(P,t)$ to obtain for $P \neq 1$, Figure 1. Cell decomposition of the torus

Figure 2. Hasse diagram of $P$. 
\[ \hat{g}(P, 1) + u \hat{h}(P, t) = \sum_{x \in P} \hat{g}(x, t)u^{|P| - \rho(x)}. \]

Hence for \( P \neq 1 \),

\[ u^{-\rho(P)}(\hat{g}(P, t) + u \hat{h}(P, t)) = \sum_{x} \hat{g}(x, t)u^{-\rho(x)}. \]

Since \( \sum_{x \in \hat{1}} \hat{g}(x)u^{-\rho(x)} = 1 \), Möbius inversion implies,

\[ \hat{g}(P, t)u^{-\rho(P)} = \mu_{P}(\hat{0}, \hat{1}) + \sum_{x \in P, x \neq \hat{0}} (\hat{g}(x, t) + u \hat{h}(x, t))u^{-\rho(x)}\mu_{P}(x, \hat{1}) \]

Since \( \hat{P} \) is semi-Eulerian, \( \mu_{P}(x, \hat{1}) = (-1)^{\rho(P) - \rho(x)} \) for \( x \neq \hat{0} \). So,

\[ \hat{g}(P) = u^{\rho(P)}\mu_{P}(\hat{0}, \hat{1}) + \sum_{x \neq \hat{0}} (\hat{g}(x, t) + u \hat{h}(x, t))(-u)^{\rho(P) - \rho(x)}. \]

**Lemma 0.4.** If \( x < \hat{1} \), then \( \hat{g}(x, t) + u \hat{h}(x, t) = t^{\rho(x)}\hat{g}(x, 1/t). \)

*Proof.* (Lemma) Let \( r + 1 = \rho(x) \). Then \( \hat{h}(x) = a_0 + a_1t + \cdots + a_rt^r \).

By definition,

\[ \hat{g}(x, t) + u \hat{h}(x, t) = (a_{r-s} - a_{r-s-1})t^{s+1} + (a_{r-s-1} - a_{r-s-2})t^{s+2} + \cdots + (a_1 - a_0)t^r + a_0t^{r+1}. \]

where \( s = \lfloor r/2 \rfloor \). When \( r \) is odd \( r = 2s + 1 \) and when \( r \) is even \( r = 2s \).

Now rewrite (4)

\[ \hat{g}(x, t) + u \hat{h}(x, t) = \]

\[ \begin{cases} \left(a_s - a_{s-1}\right)t^{s+1} + \left(a_{s-1} - a_{s-2}\right)t^{s+2} + \cdots + (a_1 - a_0)t^s + a_0t^{s+1}, & \text{r even} \\ \left(a_{s+1} - a_s\right)t^{s+1} + \left(a_s - a_{s-1}\right)t^{s+2} + \cdots + (a_1 - a_0)t^{s+1} + a_0t^{s+2} & \text{r odd}. \end{cases} \]

Since \( \hat{0} < x < \hat{1} \) and \( P \) is semi-Eulerian, \([0, x]\) is Eulerian. So we may assume by induction on the rank of \( x \) that \( a_i = a_{r-i} \)

\[ \hat{g}(x, t) + u \hat{h}(x, t) = \]

\[ \begin{cases} \left(a_s - a_{s+1}\right)t^{s+1} + \left(a_{s+1} - a_{s+2}\right)t^{s+2} + \cdots + (a_{r-1} - a_r)t^r + a_rt^{r+1}, & \text{r even} \\ \left(a_s - a_{s+1}\right)t^{s+1} + \left(a_{s+1} - a_{s+2}\right)t^{s+2} + \cdots + (a_{r-1} - a_r)t^r + a_0t^{r+1}, & \text{r odd}. \end{cases} \]

When \( r \) is odd the first term is zero since \( a_s = a_{r-s} = a_{s+1} \). Thus, in both cases \( \hat{g}(x, t) + u \hat{h}(x, t) = t^{r+1}\hat{g}(x, 1/t). \)
Now subtract \( u \hat{h}(P, t) + \hat{g}(P, t) \) from both sides of (3) and use the lemma to obtain

\[
-u\hat{h}(P, t) = u^{\rho(P)} \mu_P(\hat{0}, \hat{1}) + \sum_{\hat{0} < x < \hat{1}} t^{\rho(x)} \hat{g}(x, 1/t)(-u)^{\rho(P) - \rho(x)}
\]

\[
\Rightarrow \hat{h}(P, t) = -(u^{d})[\mu_P(\hat{0}, \hat{1}) - (-1)^{d+1}] + \sum_{x < \hat{1}} t^{\rho(x)} \hat{g}(x, 1/t)(-u)^{d - \rho(x)}
\]

\[
= -(u^{d})[\mu_P(\hat{0}, \hat{1}) - (-1)^{d+1}] + t^{d} \hat{h}(P, 1/t).
\]

Comparing like terms of the last equation gives

\[
(7) \quad \hat{h}_{d-i} - \hat{h}_{i} = (-1)^{d-i-1} \binom{d}{i} [\mu_{P}(\hat{0}, \hat{1}) - (-1)^{d+1}].
\]

When \( d \) is even \( P \) is Eulerian, so the right hand side of (7) is zero and the equality agrees with (2). If \( d \) is odd, then, since \( \mu_P(\hat{0}, \hat{1}) = \chi(\Delta_P) - 1 \) and \( (-1)^{d+1} = \chi(S^{d-1}) - 1 \), (7) also agrees with (2).

\[\square\]

**Some extra problems**

(1) For a \((d - 1)\)-dimensional simplicial complex \( \Delta \) we know that

\[
\hat{h}_{d}(F(\Delta) \cup \hat{1}) = h_{d}(\Delta) = (-1)^{d-1} \chi(\Delta) = (-1)^{d} \mu_{F(\Delta) \cup \hat{1}}(\hat{0}, \hat{1}).
\]

\[
\text{Does } \hat{h}_{d}(P) = (-1)^{d-1} \mu_{P}(\hat{0}, \hat{1}) \text{ hold for arbitrary graded } P \text{ with } \hat{0}, \hat{1}?
\]

(2) We have seen that for a simplicial complex \( \Delta \) of dimension \( d - 1 \)

\[
h_{i}(\Delta) = \hat{h}_{i}(F(\Delta) \cup \hat{1}).
\]

\[
\text{Give examples to show that for a graded poset } P \text{ with } \hat{0}, \hat{1}, h_{i}(\Delta(P)) \text{ does not determine } \hat{h}_{i}(P), \text{ nor does } \hat{h}_{i}(P) \text{ determine } h_{i}(\Delta(P)).
\]

(3) Let \( \Delta \) be a simplicial complex which as a topological space is a \((d - 1)\)-manifold with nonempty boundary \( \partial \Delta \). It is known that

\[
h_{d-i}(\Delta) - h_{i}(\Delta) = h_{i-1}(\partial \Delta) - h_{i}(\partial \Delta) + \binom{d}{i} (-1)^{d-i-1} \chi(\Delta).
\]

\[
\text{Is there a similar formula for “semi-Eulerian posets with boundary”?}
\]