1. A subgroup $H$ of a group $G$ is a characteristic subgroup of $G$ if for all automorphisms $\phi : G \to G$, $\phi(H) = H$.

(a) Prove that the center $Z$ of a group $G$ is a characteristic subgroup of $G$.

(b) The smallest normal subgroup of a group $G$ which contains all elements of the form $g h g^{-1} h^{-1}$, $g, h \in G$, is called the commutator subgroup of $G$. Prove that the commutator subgroup of $G$ is a characteristic subgroup of $G$.

2. (a) Prove that the automorphism group of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which we denote by $\text{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, is isomorphic to $S_3$. (See Hw 4 for the automorphism group.)

(b) Let $G$ and $H$ be groups and let $\theta : G \to \text{Aut}(H)$ be a homomorphism. So, for $g \in G$ and $h \in H$, $\theta(g)(h)$ is an element of $H$. Define a binary operation $\otimes$ on $G \times H$ by

$$(g_1, h_1) \otimes (g_2, h_2) = (g_1 g_2, (h_1 \theta(g_1)(h_2))).$$

Prove that $(G \times H, \otimes)$ is a group.

(c) Choose an injection $\theta : I_3 \to \text{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Using $\otimes$ from part (b), let $J$ be the group of order twelve ($I_3 \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}), \otimes$). Is $J$ isomorphic to the dihedral group of order 12?

3. Let $I$ and $J$ be ideals of a commutative ring $R$. Let $L(I, J)$ be the subset of $R$

$$\{r \in R : ra \in I \ \forall a \in J\}$$

(a) Prove that $L(I, J)$ is an ideal of $R$.

(b) Prove that $L(I, J_1 + J_2) = L(I, J_1) \cap L(I, J_2)$.

4. Let $f(x) = x^3 - 7$ and consider $f$ as a polynomial in $\mathbb{Q}[x]$. Let $E$ be a splitting field for $f$, $\mathbb{Q} \subseteq E \subseteq \mathbb{C}$.

(a) Prove that $f$ is irreducible in $\mathbb{Q}[x]$.

(b) Prove that $\omega \in E$, where $\omega$ is the third root of unity $e^{2\pi i/3}$.

(c) Prove that $\text{Gal}(E/\mathbb{Q})$ is isomorphic to $S_3$. 

1
5. Suppose $f$ and $g$ are irreducible polynomials in $\mathbb{Q}[x]$ and let $h = f \cdot g$. Assume that the degrees of $f$ and $g$ are at least two. Let $E$ be a splitting field for $h$, $\mathbb{Q} \subseteq E \subseteq \mathbb{C}$. Prove that $G = \text{Gal}(E/\mathbb{Q})$ is a nontrivial direct sum. That is, $G \cong G_1 \times G_2$, with neither $G_1$ nor $G_2$ the trivial group.