ON UPPER BOUNDS FOR POSITIVE SOLUTIONS OF SEMILINEAR EQUATIONS

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Abstract. Suppose that $E$ is a bounded domain of class $C^{2,\lambda}$ in $\mathbb{R}^d$ and $L$ is a uniformly elliptic operator in $E$. The set $\mathcal{U}$ of all positive solutions of the equation $Lu = \psi(u)$ in $E$ was investigated by a number of authors for various classes of functions $\psi$. In [DK98] we defined, for every Borel subset $\Gamma$ of $\partial E$, two such solutions $u_\Gamma \leq w_\Gamma$. We also introduced a class of solutions $u_\nu$ in 1-1 correspondence with a certain class $\mathcal{N}_G$ of $\sigma$-finite measures $\nu$ on $\partial E$. With every $u \in \mathcal{U}$ we associated a pair $(\Gamma, \nu)$ where $\Gamma$ is a Borel subset of $\partial E$ and $\nu \in \mathcal{N}_G$. We called this pair the fine boundary trace of $u$ and we denoted it $\text{tr}(u)$.

Let $u \oplus v$ stand for the maximal solution dominated by $u + v$. We say that $u$ belongs to the class $\mathcal{E}_{L,\psi}$ if the condition $\text{tr}(u) = (\Gamma, \nu)$ implies that $u \leq w_\Gamma \oplus u_\nu$ and we say that $u$ belongs to $\mathcal{E}_{L,\psi}^*$ if the condition $\text{tr}(u) = (\Gamma, \nu)$ implies that $u \geq u_\Gamma \oplus u_\nu$.

It was proved in [DK98] that, under minimal assumptions on $L$ and $\psi$, the class $\mathcal{E}_{L,\psi}^*$ contains all bounded domains. It follows from results of Mselati ([Ms02a], [Ms02b], [Ms03]) that all $E$ of the class $C^4$ belong to $\mathcal{E}_{\Delta,\psi}$ where $\Delta$ is the Laplacian and $\psi(u) = u^2$. [Mselati proved that, in his case, $u_\Gamma = w_\Gamma$ and therefore the condition $\text{tr}(u) = (\Gamma, \nu)$ implies $u = u_\Gamma \oplus u_\nu = w_\Gamma \oplus u_\nu$.]

By modifying Mselati’s arguments, we extend his result to $\psi(u) = u^\alpha$ with $1 < \alpha \leq 2$ and all bounded domains of class $C^{2,\lambda}$.

We start from proving a general localization theorem: $E \in \mathcal{E}_{L,\psi}$ under broad assumptions on $L$, $\psi$ if, for every $y \in \partial E$ there exists a domain $E' \in \mathcal{E}_{L,\psi}$ such that $E' \subset E$ and $\partial E \cap \partial E'$ contains a neighborhood of $y$ in $\partial E$.

1. Introduction

1.1. Equation $Lu = \psi(u)$. Before we state the results, we give a brief description of our setting (which is the same as in [Dy02]).

We consider a second order differential operator

\begin{equation}
Lu(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial u(x)}{\partial x_i}
\end{equation}

in a bounded smooth domain $E$. (We use this name for domains of class $C^{2,\lambda}$.) We assume that $a_{ij}(x) = a_{ji}(x)$ and that:

1.1.A. There exists a constant $\kappa > 0$ such that

\[ \sum a_{ij}(x)t_i t_j \geq \kappa \sum t_i^2 \quad \text{for all } x, t_1, \ldots, t_d \in \mathbb{R}. \]
1.1.B. All coefficients $a_{ij}(x)$ and $b_i(x)$ are Hölder continuous in $E$.

Our objective is to investigate the set $\mathcal{U}$ of all positive solutions of the equation
\begin{equation}
Lu = \psi(u) \quad \text{in } E.
\end{equation}
Here $\psi$ is a function on $E \times \mathbb{R}_+$ and $\psi(u)$ is an abbreviation for $\psi(x, u(x))$. We assume that $\psi$ satisfies conditions:

1.1.C. For every $x$, $\psi(x, \cdot)$ is convex and $\psi(x, 0) = 0$, $\psi(x, u) > 0$ for $u > 0$.

1.1.D. $\psi(x, u)$ is continuously differentiable.

1.1.E. $\psi$ is locally Lipschitz continuous in $u$ uniformly in $x$: for every $t \in \mathbb{R}_+$, there exists a constant $c_t$ such that
\[ |\psi(x, u_1) - \psi(x, u_2)| \leq c_t |u_1 - u_2| \quad \text{for all } x \in E, u_1, u_2 \in [0, t]. \]

1.1.F. There is a constant $a$ such that $\psi(x, 2u) \leq a \psi(x, u)$ for all $u$ and $x$.

1.1.G. Function $\frac{\partial \psi(x, u)}{\partial u}$ is continuously differentiable.

1.1.H. (The Keller-Osserman condition) $\int_N^\infty ds \left[ \int_0^s \psi(u) \, du \right]^{-1/2} < \infty$ for some $N > 0$.

Condition 1.1.C implies that $\psi(x, u_1) \leq \psi(x, u_2)$ for all $0 \leq u_1 < u_2$ and the Keller-Osserman condition implies the existence of a maximal element in every class $\mathcal{U}(D)$. \footnote{$\mathcal{U}(D)$ stands for the set of all positive solutions of the equation $Lu = \psi(u)$ in $D$.}

1.2. **Singular points of a solution.** We consider the tangent cone to $\mathcal{U}$ at point $u$ which we define as the set of tangent vectors $v$ to all smooth curves $u_t$ in $\mathcal{U}$ with the properties:

(a) $u_0 = u$ and $u_t \in \mathcal{U}$ for $0 \leq t < \varepsilon$;
(b) $u_t(x)$ is monotone increasing in $t$.

Condition (a) implies that $Lu_t = \psi(u_t)$ for $0 \leq t < \varepsilon$ and therefore $v(x) = \frac{\partial u_t(x)}{\partial t}|_{t=0}$ satisfies a linear equation
\begin{equation}
Lv = av
\end{equation}
where
\[ a = \psi'(u). \]

[We use abbreviation $\psi'(u)$ for $\frac{\partial \psi(x, u)}{\partial u}$.] Since $\psi$ is monotone increasing in $u$, $a(x) \geq 0$.

Condition (b) implies that $v(x) \geq 0$. There exists a function $k_a(x, y), x \in E, y \in \partial E$ (the Poisson kernel) such that every positive solution of (1.3) has an integral representation
\begin{equation}
h(x) = \int_{\partial E} k_a(x, y) \nu(dy)
\end{equation}
where $\nu$ is a finite measure on $\partial E$. If $a$ is bounded, then $k_a$ is strictly positive. However, if $a$ blows up sufficiently fast near $y \in \partial E$, then $k_a(x, y) = 0$ for all $x$. If this is the case, then we call $y$ a point of rapid growth for $a$. We say that $y \in \partial E$
is a singular point of \( u \in \mathcal{U} \) and we write \( y \in \text{SG}(u) \) if \( y \) is a point of rapid growth of \( \psi'(u) \), i.e., if \( k_{\psi'(u)}(x,y) = 0 \) for all \( x \in E \). \(^2\)

1.3. Solutions \( u_\nu, u_\Gamma \) and \( w_\Gamma \). Denote by \( \mathcal{H}(D) \) the class of positive solutions of the equation \( Lu = 0 \) in \( D \) and put \( \mathcal{H} = \mathcal{H}(E) \). (We call elements of \( \mathcal{H} \) harmonic (or \( L \)-harmonic) functions.) The formula

\[
h_\nu(x) = \int_{\partial E} k(x,y)\nu(dy)
\]

defines a 1-1 correspondence between \( \mathcal{H} \) and the set \( M(\partial E) \) of all finite measures on \( \Gamma \). (The function \( k(x,y) \) is a special case of the kernel \( k_a(x,y) \) corresponding to the value \( a = 0 \).

Put \( \nu \in \mathcal{N}_1 \) if the integral equation \(^3\)

\[
u + G\psi(u) = h_\nu
\]

has a positive solution (we denote it \( u_\nu \)). Put \( \nu \in \mathcal{N}_0 \) if there exists a monotone increasing sequence of measures \( \nu_n \in \mathcal{N}_1 \) such that \( \nu_n \uparrow \nu \). The mapping \( \nu \rightarrow u_\nu \) from \( \mathcal{N}_1 \) to \( \mathcal{U} \) can be continued to a monotone mapping from \( \mathcal{N}_0 \) to \( \mathcal{U} \). We call \( \nu \) moderate if \( \nu \in \mathcal{N}_1 \) and \( \sigma \)-moderate if \( \nu \in \mathcal{N}_0 \).

For every Borel set \( B \subset \partial E \), we define \( \mathcal{N}_1(B) \) as the class of all measures \( \nu \in \mathcal{N}_1 \) that do not charge \( B^c \). It is proved that the supremum \( u(x) \) of \( u_\mu(x) \) over all \( \mu \in \mathcal{N}_1(B) \) belongs to \( \mathcal{U} \). We denote it \( u_B \).

For every compact subset \( K \) of \( \partial E \) there exists a maximal solution \( w_K \) of the problem

\[
Lu = \psi(u) \quad \text{in } E;
\]
\[
u = 0 \quad \text{on } \partial E \setminus K.
\]

To every Borel set \( \Gamma \subset \partial E \), there corresponds a solution \( w_\Gamma \) equal to the supremum of \( w_K \) over all compact subsets \( K \) of \( \Gamma \). We have

\[
w_\Gamma \geq u_\nu \quad \text{for all } \nu \in \mathcal{N}_1(\Gamma).
\]

Let \( u, v \in \mathcal{U} \). We denote by \( u \oplus v \) the maximal solution dominated by \( u + v \). If \( u \geq v \), then \( u \ominus v \) stands for the minimal solution which dominates \( u - v \).

1.4. Main results. The fine trace \( \text{tr}(u) \) of a solution \( u \) is a pair \( (\Gamma, \nu) \) where \( \Gamma = \text{SG}(u) \) and \( \nu \) is defined on Borel sets \( B \subset \partial E \) by the formula

\[
\nu(B) = \sup\{\mu(B) : \mu \in \mathcal{N}_1, \mu(\Gamma) = 0, u_\mu \leq u\}.
\]

It is proved (see [Dy02], Section 11.7.1) that \( \nu \) is a \( \sigma \)-finite measure which belongs to \( \mathcal{N}_0 \) and that

\[
u \in \mathcal{N}_1 \quad \text{for all } \nu \in \mathcal{N}_1(\Gamma).
\]

Moreover, \( u_\Gamma \oplus u_\nu \) is the maximal \( \sigma \)-moderate solution dominated by \( u_\Gamma \).

Recall that \( E \) belongs to the class \( \mathcal{E}_{L,\psi} \) if:

\[
\text{The condition } \text{tr}(u) = (\Gamma, \nu) \text{ implies that } u \leq w_\Gamma \oplus u_\nu.
\]

We prove:

\(^2\)An equivalent probabilistic definition of singular points is given in Section 3.

\(^3\)\( G \) is Green’s operator for \( L \) in \( E \).
Theorem 1.1. The following condition is sufficient for $E$ to be in class $\mathcal{E}_{L,\psi}$:

(B) 
If $\text{tr}(u) = (\Lambda, \nu)$ and if $\Lambda \subset \Gamma \subset \partial E$ and $\nu$ is concentrated on $\Gamma$, then $u \leq w_\Gamma$.

Theorem 1.2. $E$ belongs to $\mathcal{E}_{L,\psi}$ if, for every $y \in \partial E$, there exists a domain $E' \in \mathcal{E}_{L,\psi}$ such that $E' \subset E$ and $\partial E' \cap \partial E$ contains a neighborhood of $y$ in $\partial E$.

Theorem 1.3. All bounded smooth domains belong to $\mathcal{E}_{\Delta,\psi}$ where $\Delta$ is the Laplacian and

\begin{equation}
\psi(u) = u^\alpha, \quad 1 < \alpha \leq 2.
\end{equation}

1.5. The state of the classification program. A program to describe the set $\mathcal{U}$ of all positive solutions of the equation (1.2) has been initiated by Dynkin in the earlier 1990s. Important results in this direction were obtained by Le Gall and by Marcus and Véron. All $\sigma$-moderate solutions were described by Dynkin and Kuznetsov [DK98] under the assumptions 1.1.A–1.1.H. In the Epilogue to the monograph [Dy02] the following key problem was formulated: Is every solution $\sigma$-moderate? As a step for a solution of this problem, a question was posed: Is $w_\Gamma = u_\Gamma$ for every Borel boundary set $\Gamma$? In [Ms02a], [Ms02b], [Ms03], Mselati gave positive answers to both questions in the case of the equation $\Delta u = u^2$. The present paper is inspired by his results. The main probabilistic tool of Mselati is the Brownian snake introduced by Le Gall. This tool does not work even for the equation

\begin{equation}
\Delta u = u^\alpha
\end{equation}

with $\alpha \neq 2$ and we replace it by superdiffusions.

The problem of classification of solutions $u$ by their traces $(\Gamma, \nu)$ can be split into three parts:

A. Find a lower bound in terms $u_\Gamma$ and $u_\nu$.

B. Find an upper bound in terms $w_\Gamma$ and $u_\nu$.

C. Prove that $u_\Gamma = w_\Gamma$.

Problem A was settled in [DK98]. In [Ms02a] and [Ms03] a solution of problem C precedes the investigation of B. We study B independently of C. We use extensively Mselati’s arguments which we translate into the language of superdiffusions. We also develop some new tools to replace the Hilbert space techniques and capacities $\text{Cap}_\partial$ used in [Ms02a]–[Ms03]. In Sections 2 and 3, we describe our analytic and probabilistic tools while referring for proofs, as a rule, to [Dy02]. Theorems 1.1 and 1.2 are proved in Section 4 and Theorem 1.3 is proved in Section 5.

Proof of Theorem 1.3 is based on self-similarity of the equation (1.11) which does not hold for a more general equation (1.2). Fresh ideas are needed to make the next step and to extend Theorem 1.3 to such equation.

Problem C remains open even for the equation (1.11) with $\alpha \neq 2$. This is the most challenging outstanding part of the classification problem. For $\alpha = 2$, Problem C is solved in Chapter 3 of Mselati’s memoir [Ms03]. The solution consists of two parts: upper bounds for $w_\Gamma$ in terms of a boundary capacity $\text{Cap}_\partial$ and lower bounds for certain $\sigma$-moderate solutions. Recently Kuznetsov [Ku03] succeeded in extending the first part to the general $\alpha$ (with $\text{Cap}_\partial$ replaced by the Poisson capacity). The second part still waits for its resolution.
1.6. **Acknowledgments.** I am grateful to J.-F. Le Gall and B. Mselati for stimulating discussions. I am especially indebted to S. E. Kuznetsov for a number of his suggestions incorporated into the paper (one of them is Lemma 4.2).

2. **Analytic Tools**

2.1. **Operators** $K_D, G_D$ and $V_D$. To every open subset $D$ of $E$ there correspond the Poisson operator $K_D$ and Green’s operator $G_D$. If $D$ is bounded and smooth, then, for every function $\varphi \in B(\partial D)$,

$$K_D \varphi (x) = \int_{\partial D} k_D(x, y) \varphi(y) \gamma(dy)$$

where $k_D(x, y)$ is the Poisson kernel in $D$ and $\gamma$ is the normalized surface area on $\partial D$. If $\varphi$ is continuous, then $h = K_D \varphi$ is a unique solution of the Dirichlet problem

$$Lh = 0 \quad \text{in } D,$$

$$h = \varphi \quad \text{on } \partial D.$$

We write $h = \varphi$ at $\tilde{x} \in \partial D$ if $h(x) \to \varphi(\tilde{x})$ as $x \to \tilde{x}, x \in D$. For every $f \in B(D)$,

$$G_D f(x) = \int_D g_D(x, y) f(y) \, dy.$$  

$g_D(x, y)$ is called the Green function for $L$ in $D$. If $f$ is Hölder continuous in $\bar{D}$, then $u = G_D f$ is a unique solution of the problem

$$Lu = -f \quad \text{in } D,$$

$$u = 0 \quad \text{on } \partial D.$$  

By Theorem 4.3.1 in [Dy02], if $\psi$ satisfies conditions 1.1.C and 1.1.E, then, for every $f \in bB(\bar{E})$ and for every $D \subset E$, there exists a unique solution of the equation

$$u + G_D \psi(u) = K_D f.$$  

We denote it $V_D(f)$. We have:

2.1.A. $V_D(f) \leq K_D(f)$, in particular, $V_D(c) \leq c$ for every constant $c$.

2.1.B. If $f \leq \tilde{f}$, then $V_D(f) \leq V_D(\tilde{f})$.

2.1.C. If $f_n \uparrow f$, then $V_D(f_n) \uparrow V_D(f)$.

2.1.D. For every $D$ and every Borel $f : \partial D \to [0, \infty]$, the function $u = V_D(f)$ belongs to $U(D)$. If $D$ is smooth and if $f$ is continuous in a neighborhood $U$ of $\tilde{x} \in \partial D$, then $u = f$ at $\tilde{x}$.

Properties 2.1.B–2.1.C allow us to define $V_D(f)$ for all $f \in B(\bar{D})$ by the formula

$$V_D(f) = \sup_{\ell} V_D(f \wedge \ell).$$

Equation (2.3) and conditions 2.1.B–2.1.D are preserved under this extension. The extended operators have also the properties:

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\[4\] If $f$ is a function and $B$ is a $\sigma$-algebra, then writing $f \in B$ means that $f$ is $B$-measurable and $f \geq 0$. Writing $f \in bB$ means that $F$ is, in addition, bounded.

\[5\] See Section 8.2.1 in [Dy02].
2.1.E. If $f = 0$ off a Borel set $\Gamma \subset \partial E$, then
\begin{equation}
V_E(f) \leq w_T.
\end{equation}

This follows from 2.1.D and the definition of $w_T$.

2.1.F. If $\psi$ satisfies condition 1.1.C, then
\begin{equation}
V_D(f_1 + f_2) \leq V_D(f_1) + V_D(f_2)
\end{equation}
for all $f_1, f_2 \geq 0$.

[See [Dy02], Theorem 8.2.1.]

We write $D \in E$ if $\bar{D} \subset E$. We say that a sequence $D_n$ exhausts $E$ if $D_1 \Subset D_2 \Subset \cdots \Subset D_n \Subset \cdots$ and $E$ is the union of $D_n$.

2.1.G. If $D_n$ exhaust a bounded smooth domain $D$ and if $f$ is a continuous function on $\bar{D}$, then $V_{D_n}(f(x)) \to V_D(f(x))$.

\begin{proof}
Let $\varepsilon_n \downarrow 0$. Functions $f$ and $u = V_D(f)$ are uniformly continuous in $\bar{D}$. Hence, there exists $\delta_n > 0$ such that $|f(x) - f(\tilde{x})| < \varepsilon_n$ and $|u(x) - u(\tilde{x})| < \varepsilon_n$ for $|x - \tilde{x}| < \delta_n$. Note that $\max d(x, \partial D) \to 0$. Therefore, if $n$ is sufficiently large, then, for every $x \in \partial D_n$, there exists $\tilde{x} \in \partial D$ such that $|x - \tilde{x}| < \delta_n$. We have $|f(x) - f(\tilde{x})| < \varepsilon_n$, $|u(x) - u(\tilde{x})| < \varepsilon_n$. By 2.1.D, $u(\tilde{x}) = f(\tilde{x})$. Hence
\begin{equation}
f(x) \leq u(x) + 2\varepsilon_n, \quad u(x) \leq f(x) + 2\varepsilon_n \quad \text{for all } x \in \partial D_n.
\end{equation}
By 2.1.F and 2.1.A, $V_{D_n}(f) \leq V_{D_n}(u) + 2\varepsilon_n$ and $V_{D_n}(u) \leq V_{D_n}(f) + 2\varepsilon_n$.
\end{proof}

2.2. On solutions of $Lu = \psi(u)$. The next three propositions can also be found in Section 8.2.1 of [Dy02]:

2.2.A. (Mean value property) If $u \in U(D)$, then, for every $U \Subset D$, $V_U(u) = u$ and therefore $u + G_U\psi(u) = K_Uu$.

2.2.B. (Comparison principle) Suppose $D$ is bounded. Then $u \leq v$ assuming that $u, v \in C^2(D)$,
\begin{equation}
Lu - \psi(u) \geq Lv - \psi(v) \quad \text{in } D
\end{equation}
and, for every $\tilde{x} \in \partial D$,
\begin{equation}
\limsup[u(x) - v(x)] \leq 0 \quad \text{as } x \to \tilde{x}.
\end{equation}

2.2.C. If $D$ is a bounded smooth domain and if a function $f: \partial D \to [0, \infty)$ is continuous, then $u = V_D(f)$ is a unique solution of the problem
\begin{equation}
Lu = \psi(u) \quad \text{in } D,
\end{equation}
\begin{equation}
u = f \quad \text{on } \partial D.
\end{equation}

We also need the following property proved in [Ms02a], Proposition 1.3.3 (see also [Ms03], Proposition 1.12). \footnote{A bound for partials of $u$ is used. Mselati considers the case $L = \Delta$ and he refers to Theorem 3.9 in [GT98]. For a general $L$ we can refer to Theorem 6.2 in the same book.}

2.2.D. Every sequence $u_n \in U$ contains a subsequence $u_{n_i}$ which converges uniformly on every $D \Subset E$ to an element of $U$.
2.3. On relations between solutions of $Lu = \psi(u)$ and harmonic functions.

We say that an element $u$ of $\mathcal{U}$ is a moderate solution if $u \leq h$ for some $h \in \mathcal{H}$. The formula

$$u + G\psi(u) = h$$

establishes a 1-1 correspondence between the set $\mathcal{U}_1$ of moderate solutions and a subset $\mathcal{H}_1$ of $\mathcal{H}$: $h$ is the minimal harmonic function dominating $u$, and $u$ is the maximal solution dominated by $h$. Moderate solutions can be labelled by measures $\nu \in N_1$: $u_\nu$ is the solution corresponding to $h_\nu \in N_1$ which is defined by (1.5). (The correspondence $\nu \mapsto u_\nu$ is 1-1 and monotonic.)

We need the following properties of $\mathcal{H}_1$ and $\mathcal{U}_1$.

2.3.A. If $h \in \mathcal{H}_1$ and if $h' \leq h$ belongs to $\mathcal{H}$, then $h' \in \mathcal{H}_1$. 7

2.3.B. $\mathcal{H}_1$ is a convex cone (that is it is closed under addition and under multiplication by positive numbers). 8

2.3.C. $h_\nu = 0$ on a subset $B$ of $\partial E$ if and only if $\nu(B) = 0$.

This follows from (1.5) because $k(x, y) > 0$ for all $x \in E, y \in \partial E$.

2.3.D. If $\nu \in N_1$ and $\Gamma$ is a closed subset of $\partial E$, then $u_\nu = 0$ on $O = \partial E \setminus \Gamma$ if and only if $\nu(O) = 0$.

Proof. If $\nu(O) = 0$, then $h_\nu = 0$ on $O$ by 2.3.C, and $u_\nu = 0$ on $O$ because $u_\nu \leq h_\nu$ by (2.10).

On the other hand, if $u_\nu = 0$ on $O$, then $\nu(K) = 0$ for every closed subset $K$ of $O$. Indeed, if $\eta$ is the restriction of $\nu$ to $K$, then $u_\eta = 0$ on $\Gamma$ because $\eta(\Gamma) = 0$. We also have $u_\eta \leq u_\nu = 0$ on $O$. Hence $u_\eta = 0$ on $\partial E$. The Comparison principle 2.2.B implies that $u_\eta = 0$. Therefore $\eta = 0$. \]

2.3.E. If $\nu \in N_1$ and $\Gamma$ is a closed subset of $\partial E$, then $u_\nu \leq w_\Gamma$ if and only if $\nu \in N_1(\Gamma)$. 9

Proof. If $u_\nu \leq w_\Gamma$, then $u_\nu = 0$ on $O = \partial E \setminus \Gamma$ by the definition of $w_\Gamma$ and $\nu(O) = 0$ by 2.3.D. If $\nu(O) = 0$, then $h_\nu = 0$ on $O$ by 2.3.C and $u_\nu \leq h_\nu = 0$ on $O$. Hence $u_\nu \leq w_\Gamma$ by the definition of $w_\Gamma$. \]

2.3.F. If $h \in \mathcal{H}$ and if $G\psi(h)(x) < \infty$ for some $x \in E$, then $h \in \mathcal{H}^1$.

[See Proposition 12.2.1.A in [Dy02].] 10

7 See Corollary 3.1 in [Dy02], Section 8.3.2.
8 See Theorem 8.2.3 in [Dy02].
9 Cf. Proposition 1.24 in [Ms03] or Proposition 1.3.10 in [Ms02a].
10 Proposition 12.2.1.A is stated for the case $\psi(u) = u^n$ but its proof is applicable to a general $\psi$. 
3. Probstastic tools

3.1. L-diffusion. To every operator $L$ subject to the conditions 1.1.A-1.1.B there corresponds a strong Markov process $\xi = (\xi_t, \Pi_x)$ in $E$ called $L$-diffusion. The path $\xi_t$ is defined on a random interval $[0, \zeta)$. It is continuous and its limit $\xi_\zeta$ as $t \to \zeta$ belongs to $\partial E$.

There exists a function $p_t(x, y) > 0, t > 0, x, y \in E$ (called the transition density) such that:

$$\int_E p_s(x, z) dz \cdot p_t(z, y) = p_{s+t}(x, y) \quad \text{for all } s, t > 0, x, y \in E$$

and, for every $f \in \mathcal{B}(E)$,

$$\Pi_x f(\xi_t) = \int_E p_t(x, y) f(y) \, dy.$$ 

[Writing $f \in \mathcal{B}(E)$ means that $f \geq 0$ is a Borel function on $E$.]

An $L$-diffusion has the following properties:

3.1.A. If $\tau_D$ is the first exit time of $\xi$ from a domain $D \subset E$, then, for every $f \in \mathcal{B}(\bar{E})$,

$$\Pi_x f(\xi_{\tau_D}) = K_D f(x) \quad \text{and} \quad G_D f(x) = \Pi_x \int_0^{\tau_D} f(\xi_s) \, ds.$$ 

[[Dy02], Sections 6.2.4-6.2.5,]

3.1.B. If a function $v \geq 0$ is continuous on $\bar{E}$ and satisfies the equation (1.3), then

$$v(x) = \Pi_x v(\xi_\zeta) \exp \left( - \int_0^\zeta a(\xi_s) \, ds \right).$$ 

[[Dy02], 6.3.2.A.]

3.1.C. If $D$ is a smooth subdomain of $E$ and $\tau = \tau_D$, then

$$k_D(x, y) = k(x, y) - \Pi_x 1_{\tau<\zeta} k(\xi_\tau, y) \quad \text{for all } x, y \in \partial E \cap \partial D.$$ 

[[Dy02], 6.2.5.D.]

3.2. $h$-transform. For every diffusion $\xi$, we denote by $\mathcal{F}_{\leq t}^\xi$ the $\sigma$-algebra generated by $\xi_s, s \leq t$ and by $\mathcal{F}_t^\xi$ the minimal $\sigma$-algebra which contains all $\mathcal{F}_{\leq t}^\xi$. Let $p_t(x, y)$ be the transition density of $\xi$ and let $h \in \mathcal{H}$. To every $x \in E$ there corresponds a finite measure $\Pi^h_x$ on $\mathcal{F}_t^\xi$ such that, for all $0 < t_1 < \cdots < t_n$ and every Borel sets $B_1, \ldots, B_n$,

$$\Pi^h_x \{ \xi_{t_1} \in B_1, \ldots, \xi_{t_n} \in B_n \}$$ 

$$= \int_{B_1} d z_1 \cdots \int_{B_n} d z_n \, p_{t_1}(x, z_1) p_{t_2-t_1}(z_1, z_2) \cdots p_{t_n-t_{n-1}}(z_{n-1}, z_n) h(z_n).$$

Note that $\Pi^h_x(\Omega) = h(x)$ and therefore $\Pi^h_x = \Pi^h_x / h(x)$ is a probability measure. $(\xi_t, \Pi^h_x)$ is a strong Markov process with continuous paths and with the transition density

$$p^h_t(x, y) = \frac{1}{h(x)} p_t(x, y) h(y).$$
We use the following properties of $h$-transforms.

3.2.A. If $Y \in \mathcal{F}_{\leq t}^\xi$, then

$$\Pi_x^hY = \Pi_x Y h(\xi_t).$$

[This follows immediately from (3.4).]

3.2.B. For every stopping time $\tau$ and every pre-$\tau$ positive $Y$,

$$\Pi_x^hY_{1_{\tau<\xi}} = \Pi_x Y h(\xi_\tau).$$

[See [Dy02], Lemma 7.3.1.]

3.3. **Conditional $L$-diffusion.** For every $x \in E, y \in \partial E$, we put $\Pi_x^z = \Pi_x^h$ where $h(\cdot) = k(\cdot, z)$. Put $Z = \xi_\zeta$. It follows from (2.1) and (3.1) that, for every $f \in \mathcal{B}(\partial E)$,

$$\Pi_x f(Z) = \int_{\partial E} k(x, z) \gamma(dz).$$

(3.6)

Therefore

$$\Pi_x k(y, Z) f(Z) = \int_{\partial E} k(x, z) k(y, z) \gamma(dz)$$

is symmetric in $x, y$.

**Lemma 3.1.** \(^{11}\) For every $Y \in \mathcal{F}^\xi$ and every $f \in \mathcal{B}(\partial E)$,

$$\Pi_x Y f(Z) = \Pi_x f(Z) \tilde{\Pi}^z_x Y.$$

**Proof.** It is sufficient to prove (3.8) for $Y = \tilde{Y}_{1_{\tau<\xi}}$ where $Y \in \mathcal{F}_{\leq t}^\xi$. By 3.2.A,

$$\tilde{\Pi}^z_x Y = k(x, z)^{-1} \Pi_x^z Y = k(x, z)^{-1} \Pi_x Y k(\xi_t, z).$$

Therefore the right part in (3.8) can be interpreted as

$$\int_{\Omega'} \Pi_x (d\omega') f(Z(\omega')) k(x, Z(\omega'))^{-1} \int_\Omega \tilde{\Pi}_x (d\omega) Y(\omega) k(\xi_t(\omega), Z(\omega')).$$

Fubini’s theorem and (3.7) yield that this expression is equal to

$$\int_\Omega \Pi_x (d\omega) Y(\omega) \int_{\Omega'} \Pi_x (d\omega') f(Z(\omega')) k(\xi_t(\omega), Z(\omega')) k(x, Z(\omega'))^{-1}$$

$$= \int_\Omega \Pi_x (d\omega) Y(\omega) \int_{\partial E} f(z) k(\xi_t(\omega), z) \gamma(dz) = \Pi_x Y \Pi^{\xi_t} f(Z).$$

By the Markov property of $\xi$, this is equal to the left side in (3.8).

**Lemma 3.2.** Suppose that $\xi'$ is the part of $\xi$ in a smooth subdomain $D$ of $E$ and let $\tau = \tau_D$. Put $\Pi^y_x = \Pi_x^h$ where $h(\cdot) = k_D(\cdot, y)$. We have

$$\Pi^y_x Y_{1_{\tau<\xi}} = \Pi^y_x Y_{1_{\tau<\xi}}$$

for all $x \in D, y \in \partial E \cap \partial D$ and for all $Y \in \mathcal{F}^{\xi'}$.\(^{11}\)

\(^{11}\)Property (3.8) means that $\Pi^z_x$ can be interpreted as the conditional probability distribution given that the diffusion started from $x$ exits from $E$ at point $z$. 
Proof. It is sufficient to prove (3.9) for $Y = \tilde{Y}1_{\tau<\zeta}$ where $\tilde{Y} \in \mathcal{F}_{\tilde{\tau}}^C$. By 3.2.A, (3.3), 3.2.B and Markov property of $\xi$,

\[
\Pi^Y_x = \Pi_x Y k_D(\xi_t, y) = \Pi_x Y [k(\xi_t) - \Pi_{\xi_t} 1_{\tau<\zeta} k(\xi_t, y)] \\
= \Pi_x Y k(\xi_t, y) - \Pi_x Y 1_{\tau<\zeta} k(\xi_t, y) = \Pi^Y_x Y - \Pi^Y_x Y 1_{\tau<\zeta}.
\]

\[\square\]

3.4. $(L, \psi)$-superdiffusion. Suppose that to every open set $D \subset E$ and end every $\mu \in \mathcal{M}(E)$ 12 there corresponds a random measure $(X_D, P_{\mu})$ on $E$ such that, for every $f \in \mathcal{B}(E)$,

\[
P_{\mu} e^{-\langle f, X_D \rangle} = e^{-\langle V_D(f), \mu \rangle}.
\]

We call the family $X = (X_D, P_{\mu})$ an $(L, \psi)$-superdiffusion. The existence of a $(L, \psi)$-superdiffusion is proved for a convex class of functions $\psi$ which contains $\psi(u) = u^\alpha, 1 < \alpha \leq 2$.

We put $P_x = P_{\delta_x}$ where $\delta_x$ is the unit mass concentrated at $x$. Clearly,

\[
V_D f(x) = -\log P_x e^{-\langle f, X_D \rangle}.
\]

Denote by $\mathfrak{M}$ the $\sigma$-algebra in $\mathcal{M}(E)$ generated by the functions $F(\mu) = \langle f, \mu \rangle$ with $f \in \mathcal{B}(E)$ and let $\mathcal{F}$ stand for the $\sigma$-algebra in $\Omega$ generated by $X_D$. It follows from (3.10) that $H(\mu) = P_{\mu} Y$ is $\mathfrak{M}$-measurable for every $\mathcal{F}$-measurable $Y \geq 0$. We use the following Markov property of $X$: 13

Suppose that $Y \geq 0$ is measurable with respect to the $\sigma$-algebra $\mathcal{F}_D$ generated by $X_D, D' \subset D$ and $Z \geq 0$ is measurable with respect to the $\sigma$-algebra $\mathcal{F}_D$ generated by $X_{D'}, D'' \supset D$. Then

\[
P_{\mu}(Y Z) = P_{\mu}(Y P_{X_D} Z).
\]

3.5. Stochastic boundary values. Suppose that $u \in \mathcal{B}(E)$ and

\[
\lim \langle u, X_{D_n} \rangle = Z_u \quad \text{a.s.}
\]

for every sequence $D_n$ exhausting $E$. 14 Then we say that $Z_u$ is a stochastic boundary value of $u$ and we write $Z_u = \text{SBV}(u)$. Clearly, $Z_u$ is determined by $u$ uniquely up to equivalence. We have:

3.5.A. A stochastic boundary value $Z_u$ exists for every $u \in \mathcal{U}$ and

\[
u(x) = -\log P_x e^{-Z_u}.
\]

In particular, for every $\nu \in \mathcal{N}_0$

\[
u(x) = -\log P_x e^{-Z_{\nu}}
\]

where

\[
Z_{\nu} = \text{SBV}(u_{\nu}).
\]

If $\nu \in \mathcal{N}_1$, then $P_x Z_{\nu} = h_{\nu}(x)$ and

\[
u_{\nu} + G\psi(u_{\nu}) = h_{\nu}.
\]

12 We denote by $\mathcal{M}(E)$ the set of all finite measures on $E$ and we write $(f, \mu)$ for the integral of $f$ with respect to $\mu$.

13 See [Dy02], 3.1.3.D.

14 Writing “a.s.” means “$P_\varepsilon$-a.s. for all $x \in E$.”
3.5.B. Put \( Z \in \mathcal{Z} \) if \( Z = \text{SBV}(u) \) for some \( u \in \mathcal{U} \). If \( Z_1, Z_2 \in \mathcal{Z} \), then \( Z_1 + Z_2 \in \mathcal{Z} \) and
\[
(3.15) \quad -\log P_x e^{-Z_1 - Z_2} \leq -\log P_x e^{-Z_1} - \log P_x e^{-Z_2}.
\]

3.5.C. If \( Z_1, \ldots, Z_n, \ldots \in \mathcal{Z} \) and if \( Z_n \to Z \) a.s., then \( Z \in \mathcal{Z} \).

3.5.D. If \( Z \in \mathcal{Z} \) and if \( h(x) = P_x Z \) is finite at some point \( x \in E \), then \( h \in \mathcal{H}_1 \) and \( u(x) = -\log P_x e^{-Z} \) is a moderate solution.

These properties follow from the results of Chapter 9 in [Dy02] (see Theorems 9.1.1–9.1.3 and propositions in Section 9.2.2).

We also need the following lemma.

**Lemma 3.3.** Suppose that \( f \) is a continuous function on the closure \( \bar{E} \) of a smooth domain \( E \). If \( u = V_E(f) \), then \( Z_u = \langle u, X_E \rangle \) a.s.

**Proof.** Consider a sequence \( D_n \) exhausting \( E \) and put \( Y_n = \langle u, X_D_n \rangle, Y = \langle u, X_E \rangle \). We have
\[
P_x(e^{-Y_n} - e^{-Y})^2 = P_x e^{-2Y_n} + P_x e^{-2Y} - 2P_x e^{-Y_n - Y}.
\]
By (3.10), the first two terms are equal to \( e^{-V_D_n(2u)(x)} \) and \( e^{-V_E(2u)(x)} \). By (3.13) and (3.10), the third term is equal to
\[
P_x e^{-Y_n} P_{X_D_n} e^{-Y} = P_x e^{-\langle F, X_{D_n} \rangle} = e^{-V_{D_n}(F)(x)}
\]
where \( F = f + V_E(u) \) is a continuous function on \( \bar{E} \). By 2.1.E, \( V_{D_n}(2u) \to V_E(2u) = v_1 \) and \( V_{D_n}(F) \to V_E(F) = v_2 \). Note that \( u = V_E(f) = f \) on \( \partial E \). By 2.1.D, \( v_1 = v_2 = 2f \) on \( \partial E \). By 2.2.B, \( v_1 = v_2 \) in \( E \). Hence, \( e^{-Y_n} \to e^{-Y} \) in \( L^2(P_x) \). On the other hand, \( e^{-Y_n} \to e^{-Z_n} \) \( P_x \)-a.s. Therefore \( Y = Z_u \) a.s. \( \square \)

3.6. **On solutions** \( w_T \). The range of \( X \) is a minimal closed set \( \mathcal{R} \) such that, for every \( D \subset E \) and every \( \mu \in \mathcal{M}(E) \), the measure \( X_D \) is concentrated, \( P_\mu \)-a.s., on \( \mathcal{R} \). We have
\[
(3.16) \quad w_T(x) = -\log P_x \{ \mathcal{R} \cap \Gamma = \emptyset \}.
\]
[See [Dy02], Theorem 10.1.3.]

We need the following properties of \( w_T \):

3.6.A. For every Borel set \( \Gamma \subset \partial E \), \( w_T(x) \) is equal to the infimum of \( w_\mu(x) \) over all open subsets \( O \) of \( \partial E \).

[This follows from the relations (3.1) and (3.6) in Chapter 10 of [Dy02].]

3.6.B. If \( \Gamma \subset A \cup B \), then \( w_T \leq w_A + w_B \).

[See [Dy02], 10.1.3.A and 10.1.3.E.]

3.7. **A relation between superdiffusions and conditional diffusions.** \(^{15}\) For every \( u \in \mathcal{U} \) and every \( \nu \in \mathcal{N}_0 \),
\[
(3.17) \quad P_x Z_\nu e^{-Z_u} = e^{-u(x)} \int_{\partial E} \Pi_x^y e^{-\Phi(u)} \nu(dy)
\]
where
\[
(3.18) \quad \Phi(u) = \int_0^C \psi'[u(\xi_t)] \, dt.
\]
\(^{15}\)See [Dy02], Section 9.3.
This relation can be used to prove that the condition
\[ \Pi^y_x \{ \Phi(u) < \infty \} = 0 \] for some \( x \in E \)
is sufficient and the condition
\[ \Pi^y_x \{ \Phi(u) < \infty \} = 0 \] for all \( x \in E \)
is necessary for \( y \in \partial E \) to be singular for \( u \). \[ ^{16} \text{Recall that an analytic definition of singular points was given in the Introduction.} \]

4. Proof of Theorems 1.1 and 1.2

4.1. Proof of Theorem 1.1. We need to prove that the condition (B) implies the condition (A). Let \( \text{tr}(u) = (\Gamma, \nu) \). We will prove that \( u \leq w_\Gamma \oplus u_\nu \) by applying (B) to \( v = u \ominus u_\nu \). (It follows from (1.8) that \( u_\nu \leq u \).) It is easy to check that \( v \oplus u_\nu = u \).

Let \( \text{tr}(u \ominus u_\nu) = (\Lambda, \mu) \). Clearly \( \Lambda \subset \Gamma \). If we show that \( \mu(\Gamma^c) = 0 \), then (B) will imply that \( v \leq w_\Gamma \) and therefore \( u = v \oplus u_\nu \leq w_\Gamma \oplus u_\nu \).

It remains to prove that \( \mu(\Gamma^c) = 0 \). By the definition of the trace,
\[ (4.1) \quad \mu(\Gamma^c) = \text{sup}\{ \lambda(\Gamma^c) : \lambda \in N_1, \lambda(\Gamma) = 0, u_\lambda \leq v \}. \]
If \( \lambda \in N_1, \lambda(\Gamma) = 0, u_\lambda \leq v \), then \( (\lambda + \nu)(\Gamma) = 0 \), \( \lambda + \nu \in N_1 \) by 2.3.B and \( u_{\lambda + \nu} = u_\lambda \oplus u_\nu \leq v \oplus u_\nu = u \). By (1.8), \( \lambda + \nu \leq \nu \). Hence \( \lambda = 0 \) and \( \mu(\Gamma^c) = 0 \) by (4.1).

4.2. Preparation for proving Theorem 1.2. Suppose that \( E' \) is a smooth subdomain of a bounded smooth domain \( E \). Put \( A = \partial E \cap \partial E' \) and denote by \( A_0 \) the set of \( y \in A \) such that \( d(y, \partial E' \setminus A) > 0 \).

We need the following lemmas.

Lemma 4.1. Denote by \( k' \) the Poisson kernel of \( L \) in \( E' \). If \( \nu \) is a finite measure on \( A \) and if
\[ h(x) = \int_A k(x, y) \nu(dy), \]
then
\[ h(x) = \int_{\partial E'} k'(x, y) \nu'(dy) \]
where
\[ (4.3) \quad \nu'(dy) = a(y) \nu(dy) \quad \text{on } A. \]

Proof. The restriction of \( k_y(x) = k(x, y) \) to \( E' \) is harmonic in \( E' \) and therefore it has a unique representation in the form
\[ k_y(x) = \int_{\partial E'} k'(x, z) \sigma_y(dz). \]
If \( y \in A \) and \( \Gamma \) is a closed subset of \( A \setminus \{ y \} \), then \( q(x) = \int_{\partial E'} k'(x, y) \sigma_y(dy) \leq k_y(x) \) is a harmonic function in \( E' \) vanishing on \( \partial E' \setminus \Gamma \). It vanishes also on \( \Gamma \) because so does

\[ ^{16} \text{See [Dy02], Section 11.1.3.} \]
The minimal harmonic function \( h(y) \). Hence, \( q = 0 \) which implies \( \sigma_y(\Gamma) = 0 \). We conclude that \( \sigma_y(A \setminus \{ y \}) = 0 \). By Fubini’s theorem,

\[
(4.4) \quad h(x) = \int_A \nu(dy) k_y(x) = \int_{\partial E'} k'(x, z) \sigma_z(dz) \quad \text{for } x \in E'
\]

where \( \sigma_z = \int_A \nu(dy) \sigma_y \). By comparing (4.2) with (4.4) we conclude that \( \nu' = \sigma_y \). Since \( \sigma_y \) does not charge \( A \setminus \{ y \} \), we get (4.3) with \( a(y) = \sigma_y \{ y \} \).

A class \( \mathcal{N}_1 \) of finite measures on \( \partial E \) which is in a 1-1 correspondence with the class of moderate solutions in \( E \) was introduced in Section 2.3. We denote by \( \mathcal{N}'_1 \) an analogous class for the subdomain \( E' \).

**Lemma 4.2.** Suppose that \( K \subset A \) is at a distance \( \beta > 0 \) from \( \partial E' \setminus A \) and that \( \nu \in \mathcal{M}(K) \). Let \( u \) be the maximal element of \( \mathcal{U} \) dominated by

\[
h(x) = \int_K k(x, y) \nu(dy)
\]

and let \( \tilde{u} \) be the maximal element of \( \mathcal{U}(E') \) dominated by

\[
(4.5) \quad \tilde{h}(x) = \int_K k'(x, y) \nu(dy).
\]

Then \( u \geq \tilde{u} \) on \( E' \).

**Proof.** Consider

\[
E_\varepsilon = \{ x \in E : d(x, K) > \varepsilon \}, \quad E'_\varepsilon = \{ x \in E' : d(x, K) > \varepsilon \}.
\]

If \( \varepsilon < \beta \), then \( E'_\varepsilon \subset E_\varepsilon \), \( \partial E_\varepsilon \setminus E'_\varepsilon \supset B_\varepsilon = \{ x \in E : d(x, K) = \varepsilon \} \) and \( \partial E'_\varepsilon \setminus B_\varepsilon \subset E_\varepsilon \).

Let \( u_\varepsilon = V_{E_\varepsilon}(h), \tilde{u}_\varepsilon = V_{E'_\varepsilon}(\tilde{h}) \). We have

\[
(4.6) \quad u_\varepsilon + G_{E_\varepsilon} \psi(u_\varepsilon) = h.
\]

Hence \( u_\varepsilon \leq h \). By the Mean value property 2.2.A and by 2.1.B, if \( \varepsilon > \varepsilon_2 \), then

\[
u_2 = V_{E_\varepsilon}(u_\varepsilon) \leq V_{E'_\varepsilon}(h) = u_\varepsilon
\]

and (4.6) implies that \( u_\varepsilon \downarrow u \) as \( \varepsilon \downarrow 0 \). Similarly, \( \tilde{u}_\varepsilon \downarrow \tilde{u} \). The lemma will be proved if we demonstrate that \( u_\varepsilon \geq \tilde{u}_\varepsilon \) in \( E'_\varepsilon \). This follows from the Comparison principle 2.2.B because

\[
\tilde{u}_\varepsilon = \tilde{h} \leq h = u_\varepsilon \quad \text{on } B_\varepsilon
\]

and

\[
\tilde{u}_\varepsilon = 0 \leq u_\varepsilon \quad \text{on } \partial E' \setminus B_\varepsilon.
\]

\[\square\]

**Lemma 4.3.** If a measure \( \nu \in \mathcal{N}'_1 \) is concentrated on \( A_0 \), then \( \nu \in \mathcal{N}_1 \).

**Proof.** If the restriction of \( \nu \) to any compact subset \( K \) of \( A_0 \) belongs to \( \mathcal{N}_1 \), then \( \nu \in \mathcal{N}_0 \) and, if it is finite, then \( \nu \in \mathcal{N}_1 \). Thus we can assume that \( \nu \) is concentrated on a compact set \( K \subset A_0 \). Consider functions \( h, h, u, \tilde{u} \) introduced in Lemma 4.2. The minimal harmonic function \( h^* \) dominating \( u \) has a representation

\[
h^*(x) = \int_K k(x, y) \mu(dy).
\]
By Lemma 4.1,
\begin{equation}
(4.7) \quad h^*(x) = \int_K k'(x, y) \mu'(dy)
\end{equation}
where
\[ \mu'(dy) = a(y) \mu(dy). \]

By Lemma 4.2, \( \tilde{u} \leq u \) and, since \( u \leq h^* \), we have \( \tilde{u} \leq h^* \). This implies \( \tilde{h} \leq h^* \) (because \( \tilde{h} \) is the minimal harmonic function dominating \( \tilde{u} \)). By comparing (4.5) and (4.7), we get \( \nu(dy) \leq a(y) \mu(dy) \). Since \( \mu \in N_1 \), we deduce from 2.3.A and 2.3.B that \( \nu \in N_1 \).

It follows from Lemma 4.3 that a moderate solution \( u_\eta \) in \( E \) and a moderate solution \( u'_\eta \) in \( E' \) correspond to every \( \eta \in N_1' \).

For \( C \subset A \) both functions \( w_C \in U \) and \( w'_C \in U' \) are defined. Clearly,
\begin{equation}
(4.8) \quad u'_C(x) \leq w_C(x) \quad \text{and for all } x \in E'.
\end{equation}

Lemma 4.4. If a measure \( \eta \in N_1' \) is concentrated on a closed subset \( K \) of \( A_0 \), then, for every \( y \in A_0 \),
\begin{equation}
(4.9) \quad \lim_{x \to y} [u_\eta(x) - u'_\eta(x)] = 0.
\end{equation}

Proof. It follows from 3.1.C that
\begin{equation}
(4.10) \quad h_\eta(x) = h'_\eta(x) + \Pi_x 1_{\tau < \zeta} h_\eta(\xi_\tau).
\end{equation}
This implies \( h_\eta \geq h'_\eta \) and
\begin{equation}
(4.11) \quad h_\eta(x) - h'_\eta(x) \to 0 \quad \text{as } x \to y.
\end{equation}
The equation (4.9) will be proved if we show that
\begin{equation}
(4.12) \quad 0 \leq u_\eta - u'_\eta \leq h_\eta - h'_\eta \quad \text{in } E'
\end{equation}
Note that
\begin{equation}
(4.13) \quad u_\eta + G\psi(u_\eta) = h_\eta \quad \text{in } E
\end{equation}
and
\begin{equation}
(4.14) \quad u'_\eta + G'\psi(u'_\eta) = h'_\eta \quad \text{in } E'.
\end{equation}
Hence
\begin{equation}
(4.15) \quad u_\eta - u'_\eta = h_\eta - h'_\eta - G\psi(u_\eta) + G'\psi(u'_\eta).
\end{equation}
On the other hand,
\begin{equation}
(4.16) \quad u_\eta + G'\psi(u_\eta) = \tilde{h} \quad \text{in } E'
\end{equation}
where \( \tilde{h} \) is the minimal harmonic majorant of \( u_\eta \) in \( E' \). By using the Markov property of \( \xi \), we get from (4.13) and (4.16)
\begin{equation}
(4.17) \quad \tilde{h}(x) = h_\eta(x) - \Pi_x \int_0^\xi \psi(u_\eta(\xi_s)) \, ds = h_\eta(x) - \Pi_x 1_{\tau < \zeta} G\psi(u_\eta)(\xi_\tau) \quad \text{for } x \in E'.
\end{equation}
Since \( G\psi(u_\eta) \leq h_\eta \) by (4.13), we get from (4.17) and (4.10),
\begin{equation}
\tilde{h}(x) \geq h_\eta(x) - \Pi_x 1_{\tau < \zeta} h_\eta(\xi_\tau) = h'_\eta(x) \quad \text{in } E'.
\end{equation}
which implies \( u_\eta \geq u_\eta' \) in \( E' \). Since \( \psi \) is monotonic and \( g' \leq g \), we have \( G'\psi(u_\eta) \leq G'\psi(u_\eta') \leq G\psi(u_\eta) \). Formula (4.12) follows from (4.15).

\[ \square \]

**Lemma 4.5.** Suppose that \( u' \) is the restriction of \( u \in \mathcal{U} \) to \( E' \) and let

\[ \text{tr}(u) = (\Lambda, \nu), \quad \text{tr}(u') = (\Lambda', \nu') \]

We have

\[ (4.19) \quad \Lambda' = \Lambda \cap A, \]

and \( \nu'(C) = 0 \) for every \( C \subset A_0 \) such that \( C \cap \Lambda' = \emptyset \).

\[ \text{Proof.} \ 1^\circ \ 	ext{To prove (4.19) we consider the part} \ \xi' \ 	ext{of the} \ L\text{-diffusion} \ \xi \ 	ext{in} \ E'. \ 	ext{If} \ y \in \partial E' \setminus A, \ 	ext{then,} \ \Pi^y_x\text{-a.s.,} \ u'(\xi'_t) \ 	ext{is bounded and therefore} \ \Phi(u') < \infty. \ 	ext{Hence,} \ \Lambda' \subset A. \]

For all \( x \in E', y \in A \), by Lemma 3.2,

\[ (4.20) \quad \Pi^y_x\{\Phi(u') < \infty\} = \Pi^y_x\{\Phi(u') < \infty, \tau = \zeta\} = \Pi^y_x\{\Phi(u) < \infty, \tau = \zeta\} \]

which implies \( \Lambda \cap A \subset \Lambda' \).

If \( y \in \Lambda' \), then \( y \in A \). Hence \( \Pi^y_x\{\tau \neq \zeta\} = 0 \) for all \( x \in E' \) and, by (4.20), \( \Pi^y_x\{\Phi(u) < \infty\} = 0 \). Therefore \( \Lambda' \subset A \).

\[ 2^\circ \ 	ext{Denote by} \ \mathcal{K} \ 	ext{the class of compact subsets of} \ A_0 \ 	ext{such that the restriction of} \ \nu' \ 	ext{to} \ K \ 	ext{belongs to} \ \mathcal{N}'_1. \ 	ext{To prove the second statement of the lemma, it is sufficient to prove that} \]

\[ (4.21) \quad \text{If} \ K \in \mathcal{K}, \ \eta \leq \nu' \ 	ext{and} \ \eta \ 	ext{is concentrated on} \ K, \ 	ext{then} \ \eta(C) = 0. \]

Indeed, there exist \( \mu_n \in \mathcal{N}'_1 \) such that \( \mu_n \uparrow \nu' \) and there are compact sets \( K_m \) such that \( K_m \uparrow A_0 \). Denote by \( \eta_{m,n} \) the restriction of \( \mu_n \) to \( K_m \). If (4.21) is true, then \( \eta_{m,n}(C) = 0 \) which implies that \( \nu'(C) = 0 \).

To prove (4.21) it is sufficient to establish that :

\[ (4.22) \quad \eta \in \mathcal{N}'_1, \ \eta(\Lambda) = 0, \ u_\eta \leq u. \]

Indeed, by (1.8), properties (4.22) imply that \( \eta \leq \nu \) and therefore \( \eta(C) \leq \nu(C) = 0 \).

It remains to prove (4.22).

The definition of \( \mathcal{K} \) implies that \( \eta \in \mathcal{N}'_1 \). By Lemma 4.3, \( \eta \in \mathcal{N}_1 \).

By (4.19), \( \Lambda \subset \Lambda' \cup (\partial E \setminus A) \). Hence \( \eta(\Lambda) = 0 \) because \( \eta(\Lambda') \leq \nu'(\Lambda') = 0 \) and \( \eta \) is concentrated on \( K \subset A \).

We have \( u_{\eta'} \leq u_{\nu'} \leq u_{\Lambda'} \oplus u_{\nu'} \) and therefore, by (1.9), \( u_{\eta'} \leq u \). Since \( u_\eta(x) \leq h_\eta(x) \), we have

\[ \lim_{x \to y} u_\eta(x) = 0 \leq u(x) \quad \text{for} \ y \in \partial E \setminus K. \]

By Lemma 4.4,

\[ \limsup_{x \to y} [u_\eta(x) - u(x)] = \limsup_{x \to y} [u_{\eta'}(x) - u'(x)] \leq 0 \quad \text{for} \ y \in A_0 \]

By the Comparison principle 2.2.B, this implies \( u_\eta \leq u \) in \( E \).
4.3. **Proof of Theorem 1.2.** By Theorem 1.1, it is sufficient to check condition (B). We need to prove that, if \( \text{tr}(u) = (\Lambda, \nu) \) and if \( \nu(\Gamma) = 0 \) where \( \Lambda \subset \Gamma \subset \partial E \), then \( u \leq w_\Gamma \).

The main step is to show that
\[
\limsup_{x \to y} [u(x) - 2w_\Gamma(x)] \leq 0 \quad \text{for all} \quad y \in \partial E.
\]

(4.23)

Fix \( y \) and consider a domain \( E' \in \mathcal{E}_{L,\psi} \) such that \( \partial E' \cap \partial E \) contains a neighborhood of \( y \). We use the notation introduced in Lemma 4.5. Clearly, \( y \in A_0 \). By the definition of \( \mathcal{E}_{L,\psi} \),
\[
u' \leq w_{\Lambda'} \leq w_{\Gamma'}. \tag{4.25}
\]

Note that \( \partial E' = A_0 \cup B \) where \( B \) is the closure of \( \partial E' \cap E \). By Lemma 4.5, \( \nu'(A_0 \setminus \Lambda') = 0 \). Hence \( \nu' \) is concentrated on \( \Lambda' \cup B \). By (1.7) and 3.6.B,
\[
u' \leq w_{\Lambda' \cup B} \leq w_{\Lambda'} + w_B. \tag{4.26}
\]

By (4.24), (4.25), (4.26) and (4.8),
\[
u' \leq 2w_{\Gamma} + w_B \quad \text{on} \quad E'.
\]

Therefore
\[
\limsup_{x \to y} [u(x) - 2w_\Gamma(x)] = \limsup_{x \to y} [u'(x) - 2w_\Gamma(x)] \leq \limsup_{x \to y} w_B(x) = 0.
\]

By the Comparison principle 2.2.B, (4.23) implies \( u \leq 2w_\Gamma \) in \( E \). Hence \( Z_u \leq 2Z_\Gamma \). Since \( Z_\Gamma = 0 \) or \( \infty \), we get \( Z_u \leq Z_\Gamma \). Therefore \( u \leq w_\Gamma \). \qed

5. **Proof of Theorem 1.3 for star domains**

5.1. A domain \( E \) is called a star domain relative to a point \( c \) if, for every \( x \in E \), the line segment \([c, x]\) connecting \( c \) and \( x \) is contained in \( E \). Theorem 1.3 will follow from Theorem 1.2 if we prove that all bounded smooth star domains \( E \) belong to the class \( \mathcal{E}_{\Delta, u} \) where \( u \) is given by (1.10). We relay again on Theorem 1.1. After some preparations, we demonstrate that the condition (B) in this theorem is satisfied in our case. Without loss of generality we can assume that \( c = 0 \).

We use the the self-similarity of the equation
\[
\Delta u = u^\alpha \quad \text{in} \quad E.
\]

Let \( 0 < r \leq 1 \). Put \( E_r = rE \), \( \beta = 2/(\alpha - 1) \) and
\[
f_r(x) = r^\beta f(rx) \quad \text{for} \quad x \in E, f \in \mathcal{B}(E).
\]

(5.2)

If \( u \in \mathcal{U} \), then \( u_r \) also belongs to \( \mathcal{U} \). Moreover, for \( r < 1 \), \( u_r \) is continuous on \( \bar{E} \) and \( u_r \to u \) uniformly on each \( D \subset E \) as \( r \uparrow 1 \). If \( f \) is continuous, then
\[
V_{E}(f_r)(x) = r^\beta V_{E_r}(f)(rx) \quad \text{for all} \quad x \in E.
\]

(5.3)

This is trivial for \( r = 1 \). For \( r < 1 \) this follows from 2.2.C because both parts of (5.3) are solutions of the equation (5.1) with the same boundary condition \( u = f_r \) on \( \partial E \).
5.2. Preparations.

Lemma 5.1. Put

\[ Y_r = \exp(-Z_{u_r}). \]

For every \( \gamma \geq 1 \),

\[ P_0|Y_r - Y_1|^\gamma \to 0 \quad \text{as} \ r \uparrow 1. \]

Proof. 1°. First we prove that

\[ \lim_{r \uparrow 1} P_0(Y_r^k - Y_1^k)^2 = 0 \]

for every positive integer \( k \). If (5.6) does not hold, then

\[ \lim_{r \uparrow 1} P_0(Y_r^k - Y_1^k)^2 > 0 \]

for some sequence \( r_n \uparrow 1 \).

Note that

\[ P_0(Y_r^k - Y_1^k)^2 = F_r + F_1 - 2G_r \]

where \( F_r = P_0Y_r^{2k}, G_r = P_0(Y_rY_1)^k \).

By Lemma 3.3 and (3.12),

\[ F_r = P_0 \exp[-2k(u_r, X_E)] = \exp[-V_E(2ku_r)(0)] \quad \text{for all} \ 0 < r < 1. \]

We apply (5.3) to \( f = 2ku \) and \( x = 0 \). Note that \( f_r = 2ku_r \). Therefore, by (5.3), \( V_E(2ku_r)(0) = r^\beta V_E(2ku)(0) \) and (5.9) implies that

\[ F_r = \exp[-r^\beta V_E(2ku)(0)] = \exp[-V_E(2ku)(0)]^r \]

Since \( \{u, X_{r_n}\} \to Z_u \) a.s., we have

\[ F_{r_n} \to F_1. \]

Put

\[ v_r(x) = -\log P_x(Y_rY_1)^k = -\log P_x \exp[-kZ_{u_r} - Z_u]. \]

By 3.5.B and 3.5.A, \( v_r \in U \) and \( Z_{v_r} \leq k(Z_{u_r} + Z_u) \). Therefore by 3.5.A,

\[ v_r \leq k(u_r + u) \quad \text{in} \ E. \]

By 2.2.D, we can choose a subsequence \( r_{kn} \) of the sequence \( r_n \) such that \( v_{r_{kn}} \)

converge uniformly on each \( D \subseteq E \) to an element \( v \) of \( U(E) \). By changing the notation we can assume that this subsequence coincides with the sequence \( r_n \). By (5.12), \( G_r = e^{-v_r(0)} \). By 3.5.A, \( P_0e^{-Z_v} = e^{-v(0)} \). By passing to the limit in (5.13), we get that \( v \leq 2ku \). Therefore \( P_0e^{-Z_v} \geq P_0e^{-2kZ_v} \).

By (5.10) and (5.11),

\[ P_0e^{-2k(u, X_{r_n})} \to F_1 \]

as \( r \to 1 \) along the sequence \( r_n \). Since \( \{u, X_{r_n}\} \to Z_u \), we get that \( P_0e^{-Z_v} \geq F_1 \).

Since \( v_{r_{kn}}(0) \to v(0) \), we have \( G_{r_{kn}} \to P_0e^{-Z_v} \geq F_1 \). Because of (5.8) and (5.11), this contradicts (5.7).

2°. If \( \gamma < m \), then \( (P_0|Z|^\gamma)^{1/\gamma} \leq (P_0|Z|^m)^{1/m} \). Therefore it is sufficient to prove (5.5) for even integers \( \gamma = m \). Since \( 0 \leq Y_1 \leq 1 \), the Schwarz inequality and (5.6) imply

\[ P_0|Y_rY_1^{m-k} - Y_1^m| \leq (P_1Y_1^{2(m-k)})^{1/2}[P_0(Y_r^k - Y_1^k)]^{1/2} \to 0 \quad \text{as} \ r \uparrow 1. \]
Therefore
\[
P_0|Y_r - Y_1|^m = P_0(Y_r - Y_1)^m = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} P_0(Y_r)^k Y_1^{m-k} \rightarrow \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} P_0 Y_1^m = 0.
\]

**Lemma 5.2.** For every \( \nu \in \mathcal{N}_1 \) and for all \( x \in E \),
\[
P_x Z^\nu \leq 1 + c_1 h_\nu(x)^2 + c_2 G(h_\nu^\alpha)(x)
\]
where \( c_1 = \frac{1}{2} e^{\gamma/(2 - \gamma)} \) and \( c_2 = \gamma / (\alpha - \gamma) \).

**Proof.** For every probability measure \( P \) and for every positive \( Z \)
\[
P Z^\gamma = P \int_0^Z \gamma \lambda^{\gamma-1} d\lambda = \int_0^\infty P\{Z > \lambda\} \gamma \lambda^{\gamma-1} d\lambda \leq \int_0^\infty P\{Z > \lambda\} \lambda^{\gamma-1} d\lambda.
\]
Function
\[\mathcal{E}(\lambda) = e^{-\lambda} - 1 + \lambda\]
is positive, monotone increasing and \( \mathcal{E}(1) = 1/e \). For each \( \lambda > 0 \), by Chebyshev’s inequality,
\[
P\{Z > \lambda\} = P\{Z/\lambda > 1\} = P\{\mathcal{E}(Z/\lambda) > 1/e\} \leq eq(1/\lambda)
\]
where \( q(\lambda) = PE(\lambda Z) \). By (5.15) and (5.16),
\[
P Z^\gamma \leq 1 + e \int_0^1 \gamma \lambda^{-\gamma-1} q(\lambda) d\lambda.
\]
We apply (5.17) to \( P = P_x \) and to \( Z = Z_\nu \). By 3.5.A,
\[
q(\lambda) = P_x e^{-\lambda Z_\nu} - 1 + \lambda P_x Z_\nu
= e^{-u_{\lambda \nu}(x)} - 1 + \lambda h_\nu(x) = \mathcal{E}(\lambda h_\nu) + \lambda h_\nu - u_{\lambda \nu}.
\]
Since \( \mathcal{E}(\lambda) \leq \frac{1}{2} \lambda^2 \), we have
\[
\mathcal{E}(u_{\lambda \nu}) \leq \frac{1}{2} u_{\lambda \nu}(x)^2 \leq \frac{1}{2} \lambda^2 h_\nu(x)^2.
\]
By 3.5.A,
\[
\lambda h_\nu - u_{\lambda \nu} = G(u_{\lambda \nu}^\alpha) \leq \lambda^\alpha G(h_\nu^\alpha).
\]
Formula (5.14) follows from (5.17), (5.18) (5.19) and (5.20). \( \square \)

To every \( x \in E \) there corresponds a capacity (we call it the Poisson capacity) defined on Borel subsets of \( \partial E \) by the formula
\[
\text{Cap}_x(B) = \sup_{\nu \in \mathcal{P}(B)} [G(h_\nu^\alpha)(x)]^{-1}
\]
where \( \mathcal{P}(B) \) is the class of all probability measures on \( B \). By 2.3.F, \( \nu \in \mathcal{N}_1 \) if \( G(h_\nu^\alpha)(x) < \infty \) for some \( x \). Therefore, if \( \text{Cap}_x(B) > 0 \), then
\[
\text{Cap}_x(B) = \sup_{\nu \in \mathcal{P}(B)} [G(h_\nu^\alpha)(x)]^{-1}
\]
Lemma 5.3. If \( P \subseteq \mathcal{P}(B) \), we use a bound
\[
(5.22) \quad \max_B(x) \leq C(x) \text{Cap}_x(B)^{1/(\alpha-1)}
\]
where \( C(x) \) does not depend on \( B \). For \( \alpha = 2 \), this is an implication of Theorem 3.19 in [Ms03]. \(^{17}\) For \( 1 < \alpha \leq 2 \), \((5.22)\) is proved by Kuznetsov [Ku03].

**Lemma 5.3.** If \( w_{B_n}(0) \geq \gamma > 0 \) then there exist \( \nu_n \in \mathcal{P}(B_n) \) such that \( P_0 Z_{\nu_n}^\gamma \) are bounded for every \( 1 < \gamma < \alpha \) and, consequently, \( Z_{\nu_n} \) are uniformly \( P_0 \)-integrable.

The sequence \( Z_{\nu_n} \) contains a subsequence convergent weakly in \( L^1(P_0) \). Its limit \( Z \) has the property: \( P_0 Z > 0 \) and \( u_{\nu}(x) = -\log P_x e^{-Z} \) is a moderate solution of the equation \( \Delta u = u^\alpha \) in \( \mathcal{E} \). There exists a sequence \( \hat{Z}_k \) such that \( Z_{\nu_n} \rightarrow \hat{Z}_k \) is a convex combination of a finite number of \( Z_{\nu_n} \).

**Proof.** It follows from \((5.22)\) that, if \( w_{B_n}(0) \geq \gamma \), then for all \( n \), \( \text{Cap}_0(B_n) > \delta = \lfloor \gamma/C(0) \rfloor^{\alpha-1} \). By \((5.21)\), \( G(h_{\nu_n}^\alpha)(0) < 2/\delta \) for some \( \nu_n \in \mathcal{P}(B_n) \).

There exists a constant \( c \) such that
\[
(5.23) \quad h(0) \leq c[G(h^\alpha)(0)]^{1/\alpha}
\]
for every positive harmonic function \( h \). Indeed, if the distance of 0 from \( \partial \mathcal{E} \) is equal to \( 2\rho \), then, by the mean value property,
\[
(5.24) \quad h(0) = c_1^{-1} \int_{B_\rho} h(y) \, dy \leq (c_1 c_2)^{-1} \int_{B_\rho} g(0, y) h(y) \, dy
\]
where \( B_\rho = \{ x : |x| < \rho \} \), \( c_1 \) is the volume of \( B_\rho \) and \( c_2 = \min_{x \in B_\rho} g(0, y) \) over \( B_\rho \). By Hölder’s inequality,
\[
(5.25) \quad \int_{B_\rho} g(0, y) h(y) \, dy \leq \left[ \int_{B_\rho} g(0, y) h(y) \, dy \right]^{\alpha/\alpha'} \left[ \int_{B_\rho} g(0, y) \, dy \right]^{1/\alpha'},
\]
where \( \alpha' = \alpha/(\alpha - 1) \). Hence \((5.23)\) follows from \((5.24)\) and \((5.25)\).

By \((5.23)\),
\[
(5.26) \quad \text{Cap}_0 Z_{\nu_n}(0) \leq c[G(h_{\nu_n}^\alpha)(0)]^{1/\alpha} \leq c(2/\delta)^{1/\alpha}
\]
and \((5.14)\) implies that, for every \( 1 < \gamma < \alpha \), the sequence \( P_0 Z_{\nu_n}^\gamma \) is bounded. This is sufficient for the uniform integrability of \( Z_{\nu_n} \) (see, e. g., [Me66], p.19).

By the Dunford-Pettis criterion (see, e. g., [Me66], p.20), \( Z_{\nu_n} \) contains a weakly convergent subsequence. By changing notation, we can assume that this subsequence coincide with \( Z_{\nu_n} \). The limit \( Z \) satisfies the condition \( P_0 Z > 0 \) because \( P_0 Z_{\nu_n} \rightarrow P_0 Z \).

\[
(5.27) \quad P_0 Z_{\nu_n} = \int_{\partial \mathcal{E}} k(0, y) \nu_n(dy) \geq \inf_{\partial \mathcal{E}} k(0, y) > 0.
\]

There exists a sequence \( \hat{Z}_m \) which converges to \( Z \) in \( L^1(P_0) \) norm such that each \( \hat{Z}_m \) is a convex combination of a finite number of \( Z_{\nu_n} \). (See, e. g., [Ru73], Theorem 3.13.) A subsequence \( \hat{Z}_k \) of \( \hat{Z}_m \) converges to \( P_0 \)-a.s. By Theorem 2.1 in [Dy04], all measures \( P_\nu \) are absolutely continuous with respect to \( P_0 \) on the \( \sigma \)-algebra generated by \( Z \in \mathcal{E} \). Therefore \( \hat{Z}_k \) converges to \( Z \) \( P_\nu \)-a.s. for all \( x \in \mathcal{E} \). By 3.5.C, \( \hat{Z}_k \in \mathcal{E} \) and, by 3.5.D, \( u_{\nu} \) is a moderate solution.
5.3. **Proof of condition (B).** To prove this condition we introduce a function

\[(5.26) \quad Q_r(y) = \Pi_y \exp\left(-\int_0^\zeta u_r(\xi_t)^{\alpha-1} \, dt\right)\]

and we consider, for every \(\varepsilon > 0\) and every \(0 < r < 1\), a partition of \(\partial E\) into two sets

\[(5.27) \quad A_{r,\varepsilon} = \{y \in \partial E : Q_r(y) \leq \varepsilon\} \quad \text{and} \quad B_{r,\varepsilon} = \{y \in \partial E : Q_r(y) > \varepsilon\}.

We investigate the behavior, as \(r \uparrow 1\), of functions

\[(5.28) \quad f_{r,\varepsilon} = V_E(u_r I_{r,\varepsilon}) \quad \text{and} \quad g_{r,\varepsilon} = V_E(u_r J_{r,\varepsilon}).\]

We assume, as in the condition (B), that

\[(5.29) \quad \text{tr}(u) = (\Lambda, \mu), \Lambda \subset \Gamma \subset \partial E \quad \text{and} \quad \nu \text{ is concentrated on } \Gamma\]

and we prove:

**Lemma 5.4.** For every \(\varepsilon > 0\),

\[(5.30) \quad s_\varepsilon \leq w_\Gamma \]

where

\[s_\varepsilon(x) = \limsup_{r \uparrow 1} g_{r,\varepsilon}(x).\]

**Lemma 5.5.** Fix a relatively open subset \(O\) of \(\partial E\) which contains \(\Gamma\) and put

\[C_{r,\varepsilon} = A_{r,\varepsilon} \cap (\partial E \setminus O), \quad q(\varepsilon) = \liminf_{r \downarrow 0} w_{C_{r,\varepsilon}}(0).\]

We have

\[(5.31) \quad \lim_{\varepsilon \downarrow 0} q(\varepsilon) = 0.\]

The property (B) easily follows from these two lemmas. Indeed, \(f_{r,\varepsilon}\) and \(g_{r,\varepsilon}\) belong to \(\mathcal{U}\) by 2.1.D. By 3.6.B, \(w_{A_{r,\varepsilon}} \leq w_O + w_{C_{r,\varepsilon}}\) because \(A_{r,\varepsilon} \subset O \cup C_{r,\varepsilon}\). It follows from Lemma 5.5 that

\[\liminf_{\varepsilon \to 0} \liminf_{r \uparrow 1} w_{A_{r,\varepsilon}} \leq w_O(x).\]

By 3.6.A, this implies

\[(5.32) \quad \liminf_{\varepsilon \to 0} \liminf_{r \uparrow 1} w_{A_{r,\varepsilon}} \leq w_\Gamma(x).\]

We have \(f_{r,\varepsilon} \leq u_r\) and \(g_{r,\varepsilon} \leq u_r\) by 2.1.B and

\[(5.33) \quad u_r \leq f_{r,\varepsilon} + g_{r,\varepsilon}\]

by 2.1.F. Finally, by 2.1.E,

\[(5.34) \quad f_{r,\varepsilon} \leq w_{A_{r,\varepsilon}}.\]

From the bounds (5.33), (5.34), Lemma 5.4 and (5.32) we conclude that

\[(5.35) \quad u(x) \leq \liminf_{\varepsilon \to 0} \liminf_{r \uparrow 1} w_{A_{r,\varepsilon}} + w_\Gamma(x) \leq 2w_\Gamma(x).\]

Therefore \(Z_u \leq 2Z_\Gamma\). Since \(Z_\Gamma = \infty1_{(\emptyset, \Gamma=\emptyset)}\), we get \(Z_u \leq Z_\Gamma\) and consequently \(u \leq w_\Gamma\).

It remains to prove Lemma 5.4 and Lemma 5.5.
5.4. **Proof of Lemma 5.4.** Consider harmonic functions \( h_{r,\varepsilon} = K_E(u_{r,J,\varepsilon}) \). By (2.3), \( g_{r,\varepsilon} \leq h_{r,\varepsilon} \). First, we prove that

\[
(5.36) \quad h_{r,\varepsilon}(0) \leq u(0)/\varepsilon.
\]

By applying 3.1.B to \( v = u_r \) and \( a(u) = u^{\alpha-1} \) we get

\[
u_r(y) = \Pi_y u_r(\xi)Y
\]

where

\[
y = \exp \left[ -\int_0^\gamma u_r(\xi)\alpha^{-1}d\xi \right].
\]

By (5.26) and Lemma 3.1,

\[
u_r(y) = \Pi_y u_r(\xi)\Pi_{\xi}Y = K_E(u_rQ_r)(y).
\]

Since \( \varepsilon J_{r,\varepsilon} \leq Q_r \leq 1 \), we have

\[
u_{r,\varepsilon} = K_E(\varepsilon u_{r,J,\varepsilon}) \leq K_E(u_r) = u_r
\]

and (5.36) follows because \( u_r(0) = r^\beta u(0) \leq u(0) \).

To prove that (5.30) holds at \( x \in E \), we choose a sequence \( r_n \uparrow 1 \) such that

\[
u_{r_n,\varepsilon}(x) \rightarrow s_\varepsilon(x).
\]

The bound (5.36) and well known properties of harmonic functions (see, e. g., [Dy02], 6.1.5.B and 6.1.5.C) imply that a subsequence of \( h_{r_n,\varepsilon} \) tends to an element \( h_\varepsilon \) of \( H \). By 2.2.D, this subsequence can be chosen in such a way that \( g_{r_n,\varepsilon} \rightarrow g_\varepsilon \in U \). The bounds \( g_{r,\varepsilon} \leq h_{r,\varepsilon} \), hence \( g_\varepsilon \) is a moderate solution and it is equal to \( u_\mu \) for some \( \mu \in \mathcal{N}_1 \). By the definition of the fine trace, \( \nu(B) \geq \nu'(B) \) for all \( \mu' \in \mathcal{N}_1 \) such that \( \mu'(\Lambda) = 0 \) and \( u_{\mu'} \leq u \). The restriction \( \mu' \) of \( \mu \) to \( \Gamma \) satisfies these conditions. Indeed, \( \mu' \in \mathcal{N}_1 \) by 2.3.A; \( \mu'(\Lambda) = 0 \) because \( \Lambda \subseteq \Gamma \); finally, \( u_{\mu'} \leq u_\mu = g_\varepsilon \leq u \) because \( g_{r,\varepsilon} \leq V_E(u_r) \leq u_r \) by 2.1.B and 2.2.A. We conclude that \( \mu'(\Gamma') \leq \nu(\Gamma') \) and \( \mu' = 0 \) since \( \nu(\Gamma') = 0 \). Hence \( \mu \in \mathcal{N}_1(\Gamma) \) and, by (1.7), \( u_{\mu'}(x) = g_\varepsilon(x) \leq u_\varepsilon(x) \). By (5.37), \( s_\varepsilon(x) = g_\varepsilon(x) \) which implies (5.30). \( \square \)

5.5. **Proof of Lemma 5.5.** Clearly, \( q(\varepsilon) \leq q(\varepsilon) \) for \( \varepsilon < \varepsilon \). We need to show that \( q(0+) = 0 \). Suppose that this is not true and put \( \gamma = q(0+)/2 \). Consider a sequence \( \varepsilon_n \downarrow 0 \). Since \( q(\varepsilon_n) \geq 2\gamma \), there exists a sequence \( r_n \uparrow 1 \) such that \( w_{\varepsilon_n,r_n}(0) \geq \gamma \).

We apply Lemma 5.3 to the sets \( B_n = C_{r_n,\varepsilon_n} \). A sequence \( Z_{\nu_n} \) defined in this lemma contains a weakly convergent subsequence. We redefine \( r_n \) and \( \varepsilon_n \) to make this subsequence identical with the sequence \( Z_{\nu_n} \).

Put \( u_n = u_{r_n} \). By (3.17)

\[
P_0 Z_{\nu_n} e^{-Z_{\nu_n}} = e^{-u_n(0)} \int_{\partial E} \Pi_0 e^{-\Phi(u_n)} \nu_n(dy) \leq \int_{\partial E} k(0,y) \Pi_0 e^{-\Phi(u_n)} \nu_n(dy)
\]

where \( \Phi \) is defined by (3.18). Since \( \psi'(u) = \alpha u^{\alpha-1} \geq u^{\alpha-1} \), we have

\[
\Pi_0 e^{-\Phi(u_n)} \leq Q_{\varepsilon_n}(y).
\]

Since \( \nu_n \in \mathcal{P}(B_n) \) and since \( Q_{\varepsilon_n} \leq \varepsilon \) on \( B_n \), the right side in (5.38) does not exceed

\[
\varepsilon_n \int_{\partial E} k(0,y) \nu_n(dy) = \varepsilon_n h_{\varepsilon_n}(0).
\]
Therefore
\begin{equation}
(5.39) \quad \lim_{n \to \infty} P_{0}Z_{\nu_{n}}e^{-Z_{\nu_{n}}} = 0 \quad \text{as } n \to \infty.
\end{equation}

Let $1 < \gamma < \alpha$. By Hölder’s inequality,
\[ |P_{0}Z_{\nu_{n}}(e^{-Z_{\nu}} - e^{-Z_{\nu_{n}}})| \leq (P_{0}Z_{\nu_{n}}^{\gamma})^{1/\gamma} |P_{0}||e^{-Z_{\nu}} - e^{-Z_{\nu_{n}}}|^{1/\gamma'} \]
where $\gamma' = \gamma/(\gamma - 1) > 1$. By Lemma 5.3, the first factor is bounded. By Lemma 5.1, the second factor tends to 0. Therefore
\begin{equation}
(5.40) \quad P_{0}Ze^{-Z_{\nu}} = \lim_{n \to \infty} P_{0}Z_{\nu_{n}}e^{-Z_{\nu_{n}}} = \lim_{n \to \infty} P_{0}Z_{\nu_{n}}e^{-Z_{\nu_{n}}} = 0
\end{equation}
by (5.39). Since all measures $P_{x}$ are absolutely continuous with respect to $P_{0}$ on the $\sigma$-algebra generated by $Z \in \mathcal{F}$ (Theorem 2.1 in [Dy04]), we get
\[ P_{x}Ze^{-Z_{\nu}} = 0 \quad \text{for all } x \in E. \]
Hence $Z = 0$ a.s. on $\{Z_{\nu} < \infty\}$ which implies that $P_{x}\{Z \leq Z_{\nu}\} = 1$ and $u_{\nu}(x) = -\log P_{x}e^{-Z} \leq u(x)$.

Let $F = \partial E \setminus O$. Since $\nu_{n} \in \mathcal{N}_{1}(F)$, $w_{F} \geq u_{\nu_{n}}$ by (1.7) and therefore $Z_{F} = \text{SBV}(w_{F}) \geq \text{SBV}(u_{\nu_{n}}) = Z_{\nu_{n}}$. By Lemma 5.3, there exists a sequence of $Z_{k}$ such that $Z_{k} \to Z$ a.s. and each $Z_{k}$ is a convex combination of a finite number of $Z_{\nu_{n}}$. Therefore $Z \leq Z_{F}$ a.s. and $u_{\nu} \leq w_{F}$. Since $w_{F} = 0$ on $O$, we conclude that $u_{\nu} = 0$ on $O$. Being a moderate solution, $u_{\nu} = u_{\nu}$ for some $\nu \in \mathcal{N}_{1}$ and thus $Z = Z_{\nu}$. By 2.3.3, the relation $u_{\nu} \leq w_{F}$ implies that $\nu$ is concentrated on $F$. We have $\nu \in \mathcal{N}_{1}, \nu(\Lambda) = 0$ and $u_{\nu} \leq u$. By (1.8), $\nu \leq \mu$ and, since $\mu$ is concentrated on $\Gamma$, we conclude that $\nu(\Gamma) = 0$. Hence, $\nu(\partial E) \leq \nu(O) + \nu(\Gamma) = 0$ and $Z = Z_{\nu} = 0$. This contradicts the property $P_{0}Z > 0$ which is a part of Lemma 5.3. Therefore (5.31) is true. 

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\textbf{References}


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