1) a) Since $e^{-x} > 0$ for all $x$, we have that $0 < \frac{1}{e^x + e^{-x}} < \frac{1}{e^x} = e^{-x}$.

The integral $\int_0^\infty e^{-x} \, dx$ can be evaluated:

$$
\int_0^\infty e^{-x} \, dx = \lim_{b \to \infty} \int_0^b e^{-x} \, dx = \lim_{b \to \infty} \left[-e^{-x}\right]_0^b = \lim_{b \to \infty} (1 - e^{-b}) = 1,
$$

and therefore it converges. By the Direct Comparison Test for Improper Integrals, the integral $\int_0^\infty \frac{dx}{e^x + e^{-x}}$ converges as well.

b) The integrand is not defined at $x = 0$. But since

$$
\lim_{x \to 0} \frac{x^6}{x^8 + x^2} = \lim_{x \to 0} \frac{x^4}{x^6 + 1} = 0 = 1,
$$

(the limit is finite) the integral converges.

2) a) $a_n = \frac{1 + \sin(\frac{1}{n})}{2} \cdot \frac{1}{n^2} + \frac{1 - \sin(\frac{1}{n})}{2} \cdot \frac{1}{n^4}$. We know that $\lim_{n \to \infty} \frac{1}{n} = 0$ and since $\sin x$ is a continuous function $\lim_{n \to \infty} \sin(\frac{1}{n}) = \sin 0 = 0$. Moreover, $\lim_{n \to \infty} \frac{1}{n^2} = 0 = \lim_{n \to \infty} \frac{1}{n^4}$, so $\lim_{n \to \infty} a_n = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$, so $\{a_n\}$ converges.

b) For all $n$, $|\cos n| < 1$ and $0 < \frac{|\cos n|}{n^{1.01}} < \frac{1}{n^{1.01}}$. The series $\sum_{n=1}^\infty \frac{1}{n^{1.01}}$ is a $p$-series with $p = 1.01 > 1$, therefore convergent. By the Direct Comparison Test for series with nonnegative term, $\sum_{n=1}^\infty \frac{|\cos n|}{n^{1.01}}$ converges too.

c) We note that $b_0 = x$, $b_1 = x^2$, $b_2 = (x^2)^2 = x^4$, .... The $n$-th term of the sequence $b_n = x^{2n}$. Since $\lim_{n \to \infty} 2^n = \infty$, the sequence $b_n$ converges if and only if $|x| < 1$. For these values, the limit is zero. However, if $x > 1$, the sequence still has a limit, $\infty$ (but for $x < -1$, the limit does not exist).

3) $f(x) = \sum_{n=1}^\infty (-1)^n \frac{n^2}{(n+1)^2} x^n$.

a) First we find the radius of convergence, for which we use the Ratio Test for the series with absolute values: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| x \right| \cdot \left| \frac{n^2}{(n+1)^2} \right| = |x|$. By the Ratio Test, the series converges (absolutely) if $|x| < 1$. So the radius of convergence is 1.

To check the convergence at the endpoints, we plug in 1 and $-1$ for $x$. When $x = 1$, the series becomes the alternating $p$-series, with $p = 2$ and it converges. When $x = -1$, the series is the $p$-series with $p = 2 > 1$ which also converges. It follows that the interval of convergence is $[-1, 1]$ (in fact the series converges absolutely even at the endpoints).
b) By a theorem for power series, we can differentiate the series term-by-term, so:

\[ f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{n-1}. \]

By the same theorem, we know the radius of convergence is still 1, so this new series is (absolutely) convergent for \(|x| < 1\). At the endpoints: for \(x = 1\), we get the negative of the alternating harmonic series, which is convergent, and for \(x = -1\), we get the negative of the harmonic series which is divergent. The interval of convergence is then \((-1, 1]\).

4) a) I’m assuming here we need to solve this by definition (by differentiation, that is):

\[
\begin{align*}
f(x) &= \ln(1 + x) \quad f(0) = 0 \\
f'(x) &= \frac{1}{1 + x} \quad f'(0) = 1 \\
f''(x) &= -\frac{1}{(1 + x)^2} \quad f''(0) = -1 \\
f'''(x) &= \frac{1}{(1 + x)^3} \quad f'''(0) = 2 \\
f^{(4)}(x) &= -\frac{6}{(1 + x)^4} \quad f^{(4)}(0) = -6
\end{align*}
\]

The Taylor polynomial of degree 4 is:

\[
P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.
\]

b) We are interested in an error estimate in the interval \((0, \frac{1}{2})\), so \(x > 0\). For \(x > 0\), the Maclaurin series for \(\ln(1 + x)\) is alternating:

\[\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots, \quad -1 < x \leq 1\]

and so, by the error estimation theorem for alternating series:

\[
|\ln(1 + x) - P_{n,0}(x)| \leq |(-1)^{n+1} \frac{x^{n+1}}{n + 1}| = \frac{|x|^{n+1}}{n + 1} < \frac{1}{2^{n+1}(n+1)}.
\]

since \(x\) is in the interval \((0, \frac{1}{2})\). We want \(n\) such that \(\frac{1}{2^{n+1}(n+1)} < \frac{1}{5}\). The first such \(n\) is \(n = 1\).

5) a) The Maclaurin series for \(e^{x^2} - 1\) is obtained for that of \(e^x\):

\[e^{x^2} - 1 = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots,\]

and we have (also using the Maclaurin series for \(\cos x\)):

\[
\lim_{x \to 0} \frac{\left(x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right)^2}{\left(-\frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)} = \lim_{x \to 0} \frac{x^4 + \text{higher degree terms}}{-\frac{x^4}{24} + \text{higher degree terms}}.
\]
We divide by \( x^4 \) both the numerator and the denominator and then take the limit. The answer is 24.

b) We use the Maclaurin series for \( \sin x \) and find that the limit equals:

\[
\lim_{x \to 0} \frac{x^3 - \frac{x^5}{3!} + \cdots}{x^4} = \lim_{x \to 0} \frac{\frac{1}{3!} - \frac{x^2}{5!} + \text{higher degree terms}}{x},
\]

(after dividing by \( x^3 \)). Now the numerator approaches \( \frac{1}{6} \) and the denominator 0, so the limit does not exist (note that it is not \( \infty \), since \( x \) can take positive and negative values).

6) a) A simple example is the sequence \( a_n = (-1)^n \) which diverges. The subsequence \( a_{2n} = 1 \) clearly converges.

b) \( a_n = 2 - \frac{2}{n} \) is an increasing sequences and bounded from above by 2. Since \( f(x) \) is nondecreasing, the sequence \( \{f(a_n)\} \) is also nondecreasing and bounded from above by \( f(2) \). By the Nondecreasing Sequence Theorem, \( \{f(a_n)\} \) is convergent.