Series and Integrals

On almost every Putnam exam you will be asked to evaluate integrals or infinite sums. Often the integrands or summands are quite complicated, but there is a way to simplify the problem. The “trick” is usually different in each problem, but there are few basic hints that might help you:

- problems involving complicated integrals are rarely solved directly by finding the antiderivative. Try to find a substitution or integration by parts to make the integration easier. Take advantage of any symmetries in the problem.
- Remember that there are not many series whose sum can be computed directly. Try to be able to recognize and sum those series. In particular, for $|x| < 1$ the geometric series
  \[ \sum_{n=0}^{\infty} a x^n = \frac{A}{1-x}, \]
  for all $x \in \mathbb{R}$
  \[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \]
  and for $-1 < x \leq 1$
  \[ \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}. \]
  Remember also: \[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \] and \[ \gamma = \lim_{k \to \infty} \left( \sum_{n=1}^{k} \frac{1}{n} - \ln k \right). \]

- Another useful tool to sum series are the following identities:
  \[ \sum_{i=1}^{n} a_i b_i = A_n b_n + \sum_{i=1}^{n-1} A_i (b_i - b_{i+1}), \quad \sum_{i=1}^{n} f(i) = \frac{f(1) + f(n)}{2} + \int_{1}^{n} f(x) dx + \int_{1}^{n} f'(x) (x-[x]-1/2) dx, \]
  where $A_i = a_1 + a_2 + \cdots + a_i$ and $f$ is a $C^1$ function, respectively.

- Sometimes you don’t have to compute the sum or the integral, just determine whether it converges or diverges. Here many standard techniques from calculus helps. Example:

**Problem 1.** (a) Does \( \int_{0}^{\infty} \frac{\sin x}{x} dx \) exist?
(b) Does \( \int_{\pi/2}^{\infty} \frac{\sin^2 x}{x} dx \) exist?

*Solution of (a):* The singularity of $\frac{\sin x}{x}$ at zero is only an illusion, as this function has a finite limit there. So for any $A > 0$ the integral, $\int_{0}^{A} \frac{\sin x}{x} dx$ is well defined in the standard sense (Riemann integral). Integrating by parts:
  \[ \int_{\pi/2}^{A} \frac{\sin x}{x} dx = \frac{\cos \pi/2}{\pi/2} - \frac{\cos A}{A} - \int_{\pi/2}^{A} \frac{\cos x}{x^2} dx. \]
  Now, clearly as $A \to \infty$ the right hand side converges, as $\frac{\cos A}{A} \to 0$ and the integrand $\frac{\cos x}{x^2}$ is bounded in absolute value from above by $\frac{1}{x^2}$ which is an integrable function on $[\pi/2, \infty]$. Hence the left hand side converges as well.

**Problem 2.** For $n = 1, 2, \ldots$ let
  \[ c_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}. \]
Evaluate the sum
\[ S = \sum_{n=1}^{\infty} \frac{c_n}{n(n+1)}. \]

**Problem 3.** Let \( \sum_{n=1}^{\infty} a_n \) be a convergent series and \( a_1 > a_2 > a_3 > \cdots > 0 \). Prove that
\[ \sum_{n=1}^{\infty} n(a_n - a_{n+1}) \]
converges and determine its sum.

**Problem 4.** Evaluate the sum
\[ \ln 2 - \ln 3 + \ln 4 - \ln 5 + \cdots. \]

**Problem 5.** Let
\[ a_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n-1} \frac{1}{n} - \ln 2. \]
Show that \( \sum_{n=1}^{\infty} a_n \) exists and find its sum.

**Problem 6.** Evaluate the integral
\[ I(k) = \int_{0}^{\infty} \frac{\sin kx \cos kx}{x} \, dx, \]
for positive integers \( k \).

**Problem 7.** Evaluate the limit
\[ L = \lim_{y \to 0} \frac{1}{y} \int_{0}^{\pi} \tan(y \sin x) \, dx. \]

**Problem 8.** Evaluate the integral
\[ \int_{0}^{1} \ln x \ln(1 - x) \, dx. \]

**Problem 9.** For \( a > 0, b > 0 \) evaluate the integral
\[ \int_{0}^{a} \int_{0}^{b} e^{\max(b^2x^2, a^2y^2)} \, dy \, dx. \]

**Problem 10.** Find the sum of infinite series
\[ \sum_{n=0}^{\infty} \frac{2^n}{a^{2n} + 1}, \]
where \( a > 1 \).

**Problem 11.** Let \( a_n = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}, \) \( n = 0, 1, 2, \ldots \). Does the infinite series \( \sum_{n=0}^{\infty} a_n \) converge, and if so, what is its sum.