Problems Involving Number Theory (especially modular arithmetic)

Some problems on the Putnam exam require elementary ideas from number theory. Many of these problems can be solved using modular arithmetic. If \( n \) is an integer and \( m \) is a positive integer, we can write \( n = qm + r \), where \( q \) is an integer and \( r \in \{0, 1, \ldots, m - 1\} \). We then say that \( r \) is the value of \( n \) modulo \( m \). Note that two integers \( n_1 \) and \( n_2 \) will have the same value modulo \( m \) if and only if their difference is a multiple of \( m \). We then write \( n_1 \equiv n_2 \pmod{m} \). For example, \( 11 \equiv 7 \pmod{4} \).

Recall that if \( n_1, \ldots, n_k \) are integers, then \( \gcd(n_1, \ldots, n_k) \) is the largest integer that divides all of \( n_1, \ldots, n_k \). If \( \gcd(n_1, n_2) = 1 \), we say \( n_1 \) and \( n_2 \) are relatively prime.

A result worth remembering is the following:

**Fermat’s Little Theorem:** If \( a \) is an integer and \( p \) is a prime number that does not divide \( a \), then \( a^{p-1} \equiv 1 \pmod{p} \).

A generalization of this result can be obtained using the Euler phi-function. Given an integer \( n \), define \( \phi(n) \) to be the number of integers in \( \{1, \ldots, n\} \) that are relatively prime to \( n \). If the prime factorization of \( n \) is \( n = p_1^{a_1} \cdots p_k^{a_k} \), where \( p_1, \ldots, p_k \) are prime, then

\[
\phi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1).
\]

The Euler phi-function can be used to generalize Fermat’s Little Theorem:

**Euler’s Theorem:** If \( a \) is an integer that is relatively prime to \( n \), then \( a^{\phi(n)} \equiv 1 \pmod{n} \).

**Example:** Suppose \( a + b \equiv c + d \pmod{100} \) and \( 2^a + 2^b \equiv 2^c + 2^d \pmod{101} \). Prove that either \( 2^a \equiv 2^c \pmod{101} \) or \( 2^a \equiv 2^d \pmod{101} \).

**Solution:** By Fermat’s Little Theorem, we have \( 2^{100n} \equiv 1 \pmod{101} \) for all positive integers \( n \). Therefore, the fact that \( a + b \equiv c + d \pmod{100} \) implies that \( 2^{a+b} \equiv 2^{c+d} \pmod{101} \). Therefore, \( 2^{2c} \equiv 2^c \cdot 2^d \pmod{100} \), and so \( 2^a(2^c + 2^d - 2^c) \equiv 2^c \cdot 2^d \pmod{100} \). Equivalently, \( (2^a - 2^c)(2^a - 2^d) \equiv 0 \pmod{101} \). Since 101 is prime, it follows that either \( 2^a \equiv 2^c \pmod{101} \) or \( 2^a \equiv 2^d \pmod{101} \).

Some Practice Problems

Below are some practice problems, all of which can be solved using ideas from number theory. Modular arithmetic is helpful in several of the problems.

**Problem 1:** Let \( S \) be a set of 87 objects. How many subsets of \( S \) contain an odd number of elements?

**Problem 2:** Prove that \( n^{33} - n \) is a multiple of 15 for all positive integers \( n \).

**Problem 3:** Find all 14-tuples of nonnegative integers \((n_1, \ldots, n_{14})\), if any, such that \( n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599 \).
Problem 4: Let $n$ be a positive integer that is not divisible by 2 or 5. Prove that some multiple of $n$ has a decimal representation that consists only of ones.

Problem 5: Given a positive integer $n$, let $p(n)$ be the product of the nonzero digits of $n$. Calculate $p(1) + p(2) + p(3) + \cdots + p(10^{16})$.

Problem 6: Suppose $m$ and $n$ are positive integers and $m$ is odd. Find $\gcd(2^m - 1, 2^n + 1)$.

Problem 7: Every positive integer can be expressed as a sum of one or several consecutive positive integers. For each positive integer $n$, show that the total number of expressions of $n$ as such a sum is the number of odd divisors of $n$.

Problem 8: For a positive integer $n$, let $d(n)$ denote the number of positive integer divisors of $n$, including $n$ itself, and let $\phi(n)$ denote the number of positive integers less than or equal to $n$ that are relatively prime to $n$. Find all positive integers $n$ such that $d(n) + \phi(n) = n$. 