Basic Geometry Tools

- Problems with affine invariance often can be done using vector geometry. Orthogonality can be expressed via dot-product, the center of mass is just an average.

- Planar problems sometimes can be done using complex numbers. In particular, multiplication by a complex of norm 1 is a rotation around the origin.

- Vectors are translations and complex numbers specify rotations, possibly with a stretch. More generally, you will want to watch out for transformations, either isometries or similarities.

- Fundamental extremal properties: the shortest path between two points is a straight line, the maximal area a loop of given length can contain is the circle.

- Think of choosing extreme points or configurations. Recall that every continuous function has a max and a min on any compact (closed+bounded) set. Example: In any closed and bounded set, there is a pair of points realizing the maximum distance.

- Beware: sometimes old fashioned Euclidean geometry is all you need; sometimes the geometric wording just hides a problem whose true nature is combinatorial or that belongs to Calculus.

Example. Suppose a non-empty compact subset of the plane contains, for any pair of points, a bridging hemi-circle. Show that the subset is a disc.

Idea: take two points of maximum distance. This gives you a hemi-circle. Now consider the arcs you get from the points on that hemi-circle. This yields almost all of the disc, the final arc is included by compactness.

Example. Given an acute triangle $ABC$, let $P$ be the point that minimizes the sum of distances to the vertices of the triangle. Show that every edge of the triangle appears as a visual angle of $120^\circ$ from $P$.

We want to apply that shortest paths are straight. So we have to find a way of aligning the three segments $PA$, $PB$, $PC$. This is done by a $60^\circ$-rotation:
Thus, the solution is obtained as follows: Over each side, draw an equilateral triangle. Connect the opposite triangle vertices. The point of intersection of the connecting lines is the minimizing point.

**Problem 1.** All four edges of a (non-planar) quadrilateral in 3-space touch the unit sphere. Prove that the four touching points lie in a plane.

**Problem 2.** Let \( r_A, r_B, \) and \( r_C \) be 120°-rotations around the centers \( A, B, \) and \( C. \) Show that the composition \( r_A r_B r_C \) is a translation. Moreover, it is the identity if and only if \( ABC \) is an equilateral triangle.

**Problem 3 (Napoleon).** Given a triangle, erect equilateral triangles on all its edges. Show that the centers of these three equilateral triangles form an equilateral triangle themselves.

**Problem 4.** Show that every convex compact region contains a triangle that accounts for at least 25% of its area.

**Problem 5.** Improve the bound in the previous problem to 30%.

**Problem 6.** Improve the bound in the previous problem to its optimum: the ratio of an equilateral triangle and its enclosing circle. (Beware: this is way too hard, even for a Puntam problem.)

**Problem 7.** Consider \( n \) vectors \( v_1, \ldots, v_n \) in the plane of length \( \leq 1. \) Show that one can choose signs \( c_i \in \{1, -1\} \) such that \( c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \) has length \( \leq \sqrt{2}. \)

**Problem 8.** Fix two points \( A \) and \( B. \) Describe the set of all points that are obtained by reflecting \( A \) about a line through \( B. \)

**Problem 9.** How many solid balls does it take to shield a point source of light? (Also, show that you can do it with six balls of equal size.)

**Problem 10 (Sylvester).** Let \( M \) be a finite set of points such that for any two of them, there is a third point in \( M \) on the same line. Show that \( M \) is contained in a single straight line.

**Problem 11.** Given a triangle, determine the point that minimizes the total distance (sum) to the sides.