Maximization Problems and Inequalities

On the Putnam exam, you may be asked to find the maximum value of some function $f(x)$. It is important that your justification be rigorous. Solving the equation $f'(x) = 0$ is not sufficient because you may find a minimum or a local maximum, rather than a global maximum, if the function does not have a global maximum or attains its global maximum on a boundary point of the domain. Here are a few suggestions for making your argument rigorous:

- Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and differentiable. If $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$, then $f$ attains its maximum value at $x_0$.

- Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Then a maximum value for $f$ must exist. If $f$ is differentiable and $f$ attains its maximum at some point $x_0 \in (a, b)$, then $f'(x_0) = 0$. Therefore, if you can show that $x_0$ is the only point in $(a, b)$ at which $f'(x_0) = 0$, and also that $f(x_0) > f(a)$ and $f(x_0) > f(b)$, then $f$ must attain its maximum at $x_0$. The same idea works in higher dimensions, but the boundary of the domain is more complicated.

- Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and differentiable. Also, suppose you can show that $f(x) \leq M$ for all $x \not\in [a, b]$ and $f(x) > M$ for some $x \in (a, b)$. Then, $f$ must attain a maximum at some $x_0 \in (a, b)$, and $f'(x_0) = 0$.

Inequalities can sometimes be converted into maximization problems because one way of proving that $f(x) \leq M$ for all $x$ is to find the maximum value of $f(x)$ and then show it is less than or equal to $M$. However, not all inequalities are best approached in this way. Below are some useful inequalities:

- Don’t forget about the most important inequality: $x^2 \geq 0$ for all $x \in \mathbb{R}$. A good way to show that an expression is nonnegative is to write it as the square of another expression (or a square plus something nonnegative, or a sum of squares).

- The Cauchy-Schwartz Inequality states that if $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and $(b_1, \ldots, b_n) \in \mathbb{R}^n$, then
  $$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$ 

- Jensen’s Inequality states that if $f$ is a convex function, then
  $$\frac{f(a_1) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + \cdots + a_n}{n}\right).$$

- The arithmetic-geometric mean inequality states that if $a_1, \ldots, a_n$ are nonnegative real numbers, then
  $$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \ldots a_n)^{1/n}.$$ 

On the back of the page are ten problems for you to work on. You are invited to write up your solution to one of these problems. If you do so, I will return it to you with feedback. You can either hand in your solution at the next meeting, or give it to me in my office (Malott 409).
Problem 1. Show that the inequality
\[ x_1^2 + x_2^2 + \cdots + x_n^2 \geq (r_1 x_1 + \cdots + r_n x_n)^2 \]
holds for all real \( x_1, \ldots, x_n \) if and only if \( r_1^2 + \cdots + r_n^2 \leq 1 \).

Problem 2. Prove that if \( x_1, x_2, \ldots, x_n \) are positive numbers such that \( x_1 + \cdots + x_n = 1 \), then
\[ \sum_{i=1}^{n} \frac{1}{x_i} \geq n^2. \]

Problem 3. Suppose \( x \) and \( y \) are real numbers with \( y \geq 0 \) and \( y(y + 1) \leq (x + 1)^2 \). Prove that \( y(y - 1) \leq x^2 \).

Problem 4. Let \( x_i > 0 \) for \( i = 1, 2, \ldots, n \). For each nonnegative integer \( k \), prove that
\[ \frac{x_1^k + \cdots + x_n^k}{n} \leq \frac{x_1^{k+1} + \cdots + x_n^{k+1}}{x_1 + \cdots + x_n}. \]

Problem 5. Find the minimum value of
\[ \frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)} \]
for \( x > 0 \).

Problem 6. Find the maximum value of \( f(x) = x^3 - 3x \) on the set of all real numbers \( x \) satisfying \( x^4 + 36 \leq 13x^2 \).

Problem 7. Given that \( \{x_1, \ldots, x_n\} = \{1, \ldots, n\} \), find the largest possible value, as a function of \( n \) (with \( n \geq 2 \)) of
\[ x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1. \]

Problem 8. Find the maximum value of
\[ \int_{0}^{y} \sqrt{x^4 + (y - y^2)^2} \, dx \]
for \( 0 \leq y \leq 1 \).

Problem 9. Determine the maximum value of the sum
\[ \sum_{i<j} x_i x_j (x_i + x_j) \]
over all \( n \)-tuples satisfying \( x_i \geq 0 \) and \( x_1 + \cdots + x_n = 1 \).

Problem 10. If \( a, b, c, \) and \( d \) lie between 0 and 1, show that \( (1 - a)(1 - b)(1 - c)(1 - d) > 1 - a - b - c - d \).