Series and Integrals

On the Putnam exam, you are sometimes asked to evaluate integrals or infinite sums. Often, the integrands or summands are quite complicated. However, there is usually a procedure for simplifying the problem. While the “trick” is different in each problem, there are a few ideas worth keeping in mind:

- On problems involving complicated integrals, rarely is trying to find an antiderivative of the integrand directly the best approach to the problem. Usually, you can find a substitution that will make the integration easier. Try to take advantage of any symmetry in the problem.

- It is important to remember the formula for the sum of a geometric series. If $A$ is a real number and $-1 < r < 1$, then $\sum_{n=0}^{\infty} Ar^n = A/(1 - r)$.

- It may also be helpful to recognize the power series representations of certain functions. For all real numbers $x$, recall that
  
  $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$
  
  $\sin(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
  
  $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

  Also, if $0 < x \leq 2$, then
  
  $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x - 1)^n}{n}$.

You may also be asked to determine whether a series or an improper integral converges or diverges. Any of the standard techniques that you learned in calculus could be useful here.

Below are some more problems for you to work on. You are invited to write up your solution to one of these problems. If you do so, I will return it to you with feedback. You can either hand in your solution at the next meeting, or give it to me in my office (Malott 409).

**Problem 1.** Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

**Problem 2.** Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

**Problem 3.** Evaluate $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 4^{-(m^2+2mn+n^2+m+3n)}$.

**Problem 4.** Suppose $0 < s < 1$. Evaluate

$$\lim_{N \to \infty} \left( \sum_{k=1}^{\lfloor Ns \rfloor} \frac{1}{k} - \sum_{k=1}^{N} \frac{1 - (1 - s)^{k+1}}{k+1} \right),$$

where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to $r$.

(OVER)
Problem 5. Does \( \int_{\pi}^{\infty} \frac{\sin x}{x} \, dx \) exist? Does \( \int_{\pi}^{\infty} \frac{\sin^2 x}{x} \, dx \) exist? Prove your answers.

Problem 6. Evaluate
\[ \int_{2}^{4} \frac{\sqrt{\ln(9 - x)}}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}} \, dx. \]

Problem 7. Let \( a > 0 \) and \( b > 0 \). Evaluate
\[ \int_{0}^{a} \int_{0}^{b} e^{\max\{b^2x^2, a^2y^2\}} \, dy \, dx. \]