Recall that the *greatest common divisor* of two integers $a$ and $b$, denoted by $\gcd(a, b)$, is the greatest integer that divides $a$ and $b$. It is computed by the *Euclidean algorithm* using *division with remainder*.

**Congruences:** We write

$$a \equiv b \pmod{m}$$

if

$$m \mid (a - b).$$

For any polynomial with integer coefficients $f(t)$ this implies

$$f(a) \equiv f(b).$$

**Fermat’s little theorem:** Let $a$ be an integer and $p$ be a prime. Then

$$a^p \equiv a \pmod{p}.$$

A commonly used fact is that for any prime $p$ and $k$ such that $1 < k < p$

$$p \mid \binom{p}{k}.$$

The *Euler function* is defined by:

$$\phi(n) = p_1^{k_1 - 1}(p_1 - 1) \cdots p_i^{k_i - 1}(p_i - 1)$$

if

$$n = p_1^{k_1} \cdots p_i^{k_i}$$

for distinct primes $p_1, \ldots, p_i$.

**Euler theorem:** For any positive integers $a$ and $n$ such that

$$\gcd(a, n) = 1$$

we have

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

**Function integer part:** $\lfloor x \rfloor$ is the greatest integer $\leq x$.

Recall from *Linear Algebra* the relations between matrices and linear operators, definitions of kernel and image, relations to solutions of systems of linear equations.

Recall the definition of eigenvalues of matrices/linear operators, characteristic polynomial, trace and determinant.

Recall the *Jordan normal form* of matrices/linear operators.

**Examples:**

**Problem 1.** If $p$ is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

is
of binomial coefficients is divisible by $p^2$.

**Problem 2.** For any square matrix $A$, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$  

Prove or disprove: there exists a $2 \times 2$ matrix $A$ with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$  

**Problem 3.** For each positive integer $n$, write the sum $\sum_{m=1}^{n} 1/m$ in the form $p^n/q^n$, where $p_n$ and $q_n$ are relatively prime positive integers. Determine all $n$ such that $5$ does not divide $q_n$.

**Problem 4.** Prove that for $n \geq 2$,

$$2^{n-1} \equiv 2^{n-1} \pmod{n}.$$  

**Problem 5.** Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number $A_n$ is defined by concatenating the decimal expansions of $A_{n-1}$ and $A_{n-2}$ from left to right. For example $A_3 = A_2 A_1 = 10$, $A_4 = A_3 A_2 = 101$, $A_5 = A_4 A_3 = 10110$, and so forth. Determine all $n$ such that $11$ divides $A_n$.

**Problem 6.** Find necessary and sufficient conditions on positive integers $m$ and $n$ so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$  

**Problem 7.** For a positive real number $\alpha$, define

$$S(\alpha) = \{ \lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots \}.$$  

Prove that $\{1, 2, 3, \ldots \}$ cannot be expressed as the disjoint union of three sets $S(\alpha), S(\beta)$ and $S(\gamma)$. [As usual, $\lfloor x \rfloor$ is the greatest integer $\leq x$.]

**Problem 8.** What is the units (i.e., rightmost) digit of

$$\frac{10^{20000}}{10^{100} + 3}?$$  

**Problem 9.** A *transversal* of an $n \times n$ matrix $A$ consists of $n$ entries of $A$, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices $A$ satisfying the following two conditions:

(a) Each entry $\alpha_{i,j}$ of $A$ is in the set $\{-1, 0, 1\}$.

(b) The sum of the $n$ entries of a transversal is the same for all transversals of $A$.

An example of such a matrix $A$ is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
Determine with proof a formula for \( f(n) \) of the form
\[
f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,
\]
where the \( a_i \)'s and \( b_i \)'s are rational numbers.

**Problem 10.** For positive integers \( n \), let \( M_n \) be the \( 2n+1 \) by \( 2n+1 \) skew-symmetric matrix for which each entry in the first \( n \) subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of \( M_n \). (According to one definition, the rank of a matrix is the largest \( k \) such that there is a \( k \times k \) submatrix with nonzero determinant.)

One may note that
\[
M_1 = \begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{pmatrix},
\]
\[
M_2 = \begin{pmatrix}
0 & -1 & -1 & 1 & 1 \\
1 & 0 & -1 & -1 & 1 \\
1 & 1 & 0 & -1 & -1 \\
-1 & 1 & 1 & 0 & -1 \\
-1 & -1 & 1 & 1 & 0
\end{pmatrix}.
\]

**Problem 11.** Let \( D_n \) denote the value of the \((n - 1) \times (n - 1)\) determinant
\[
\begin{bmatrix}
3 & 1 & 1 & 1 & \cdots & 1 \\
1 & 4 & 1 & 1 & \cdots & 1 \\
1 & 1 & 5 & 1 & \cdots & 1 \\
1 & 1 & 1 & 6 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & n + 1
\end{bmatrix}.
\]

Is the set \( \{ \frac{D_n}{n!} \}_{n \geq 2} \) bounded?

Let \( \mathcal{M} \) be a set of real \( n \times n \) matrices such that
(i) \( I \in \mathcal{M} \), where \( I \) is the \( n \times n \) identity matrix;
(ii) if \( A \in \mathcal{M} \) and \( B \in \mathcal{M} \), then either \( AB \in \mathcal{M} \) or \( -AB \in \mathcal{M} \), but not both;
(iii) if \( A \in \mathcal{M} \) and \( B \in \mathcal{M} \), then either \( AB = BA \) or \( AB = -BA \);
(iv) if \( A \in \mathcal{M} \) and \( A \neq I \), there is at least one \( B \in \mathcal{M} \) such that \( AB = -BA \).

Prove that \( \mathcal{M} \) contains at most \( n^2 \) matrices.

**Problem 12.** Let \( A \) and \( B \) be \( 2 \times 2 \) matrices with integer entries such that each of \( A, A + B, A + 2B, A + 3B, A + 4B \) has an inverse with integer entries. Prove that the same must be true of \( A + 5B \).