Exercise 1 from Sections 12 & 13, page 83
We will show that $A$ is open by exhibiting it as a union of open sets. For each $x \in A$, let $U_x$ be the open set containing $x$ such that $U_x \subset A$. It is easy to see that $A = \bigcup_{x \in A} U_x$, so $A$ is open. □

Exercise 4 from Sections 12 & 13, page 83
(a) Let $T = \bigcap \mathcal{T}_\alpha$. To show that $T$ is a topology, we have to verify that $T$ satisfies the three properties in the definition of a topology:

(1) Are $\emptyset$ and $X$ in $T$? Yes, because $\emptyset$ and $X$ are in $\mathcal{T}_\alpha$ for each $\alpha$.

(2) Let $\{U_\beta\}$ be a collection of open sets in $T$. Since $T$ is the intersection of the topologies $\mathcal{T}_\alpha$, $\{U_\beta\}$ is a collection of open sets in $\mathcal{T}_\alpha$ for each $\alpha$. Hence their union $\bigcup U_\beta$ is in $\mathcal{T}_\alpha$ for each $\alpha$, and so $\bigcup U_\beta \in T$.

(3) Starting with a finite collection of open sets in $T$, the argument is as in (2) above.

This proof works for the intersection $\bigcap \mathcal{T}_\alpha$ because subsets in the intersection are open in each $\mathcal{T}_\alpha$. For the union of even two topologies, say $\mathcal{T}_1$ and $\mathcal{T}_2$, we can have subsets $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$ such that $U_1 \cup U_2$ is not in either $\mathcal{T}_1$ or $\mathcal{T}_2$. A simple example is furnished by the topologies $\mathcal{T}_1 = \{\emptyset, X, \{b\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$ on the three point set $X = \{a, b, c\}$. Taking $U_1 = \{b\}$ and $U_2 = \{a\}$, their union $\{a, b\}$ is not in either of $\mathcal{T}_1$ or $\mathcal{T}_2$.

(c) Let $\mathcal{T}$ be the topology on $X$ generated by the subbasis $\mathcal{T}_1 \cup \mathcal{T}_2$. It is easily checked that $\mathcal{T}$ is the smallest topology containing both $\mathcal{T}_1$ and $\mathcal{T}_2$. In this particular case,

$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$

and the only other subset that can be generated by taking unions of finite intersections is $\{b\}$. Hence the smallest topology containing both $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$

The largest topology contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ is clearly their intersection, so it is

$\mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, X, \{a\}\}$

in this case.
Exercise 8 from Sections 12 & 13, page 83

(a) Let $U$ be an open set in $\mathbb{R}$ and $x$ an element in $U$. By the definition of the standard topology on $\mathbb{R}$, there are real numbers $r, s$ such that $x \in (r, s) \subset U$. Since the rationals are dense in $\mathbb{R}$, we can find rational numbers $a, b$ in $(r, x)$ and $(x, s)$, respectively. This gives $x \in (a, b) \subset U$. In other words, given $U$ open in $\mathbb{R}$ and $x \in U$, we can find an $(a, b) \in \mathcal{B}$ such that $x \in (a, b) \subset U$. Hence, by Lemma 13.2, $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$.

(b) Given $x \in \mathbb{R}$, there are certainly rational numbers $a, b$ such that $a < x < b$, so $x \in [a, b)$. Thus $\mathbb{C}$ satisfies condition (1) for a basis. For condition (2), simply note that the intersection of two intervals of the form $(a, b)$ is either empty or another interval of the same form. Hence $\mathbb{C}$ is a basis for a topology on $\mathbb{R}$; call this topology $\mathcal{T}$.

Each element of $\mathbb{C}$ is open in the lower limit topology, so $\mathcal{T}$ is contained in the lower limit topology. To see that they are different, consider the open set $[r, s) \in \mathbb{R}_l$, where $r$ is irrational. If $[r, s)$ were open in $\mathcal{T}$, then, since $\mathbb{C}$ is a basis for $\mathcal{T}$ and $r \in [r, s)$, there must be $a, b \in \mathbb{Q}$ such that $r \in [a, b) \subset [r, s)$. This is clearly impossible for rational $a$, so $[r, s)$ cannot be in $\mathcal{T}$.