1. Find the point on the line $y = 2x + 5$ that is closest to the origin.

**Solution:** See the picture below.

Recall that the formula for the distance from the point $(x, y)$ to the origin is $z = \sqrt{x^2 + y^2}$. So, in our case, $z = \sqrt{x^2 + (2x + 5)^2} = \sqrt{5x^2 + 20x + 25}$. Then $z' = \frac{10x + 20}{2\sqrt{5x^2 + 20x + 25}}$. Then the only critical point for $z$ is at $x = -2$, i.e. where $z' = 0$. Observe that the shortest distance to the origin must be for some $x$ in between the $y$-axis intercept and the $x$-axis intercept, that is, for some $x \in \left[-\frac{5}{2}, 0\right]$. Let us apply the Closed Interval Method to find the absolute minimum:

- $z\left(-\frac{5}{2}\right) = \sqrt{5 \cdot \frac{25}{4} - 20 \cdot \frac{5}{2} + 25} = \frac{5}{2}$
- $z(-2) = \sqrt{5 \cdot 4 - 20 \cdot 2 + 25} = \sqrt{5}$
- $z(0) = \sqrt{5 \cdot 0 + 20 \cdot 0 + 25} = 5$

Thus, the absolute minimum is achieved at $x = -2$, $y = 1$.

2. Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius 4, so that one side of the rectangle sits on the diameter of the semicircle.

**Solution:** See picture below.
Then \( x^2 + y^2 = 4^2 \), and the area of the rectangle is \( A = (2x)y = 2x\sqrt{16 - x^2} \).

\[
A' = 2\sqrt{16 - x^2} + 2x \cdot \frac{1}{2\sqrt{16 - x^2}} \cdot (-2x) = \frac{2(16 - x^2) - 4x^2}{\sqrt{16 - x^2}} = \frac{32 - 6x^2}{\sqrt{16 - x^2}}
\]

Note that \( 0 \leq x \leq 4 \). So the critical points that we get are \( x = 4 \) and \( x = \sqrt{\frac{16}{3}} \). Using the Closed Interval Method:

- \( A(0) = 0 \)
- \( A \left( \sqrt{\frac{16}{3}} \right) = 2\sqrt{\frac{16}{3}} \sqrt{\frac{32}{3}} = \frac{32\sqrt{2}}{3} \)
- \( A(4) = 0 \)

Therefore, we get an absolute max for the rectangle with base \( \sqrt{\frac{64}{3}} \) and height \( \sqrt{\frac{32}{3}} \).

3. You are given a 4ft long wire. Construct a square and/or a circle so that the total enclosed area is:

(a) maximum  
(b) minimum

**Solution:** Let \( x \) be the side of the square, and \( r \) be the radius of the circle. If \( P_{circle} \) is the circumference of the circle, and \( P_{square} \) is the perimeter of the square, we know that the \( P_{circle} + P_{square} = 2\pi r + 4x = 4 \) and the total enclosed area is \( A = \pi r^2 + x^2 \). Let us solve for \( r = \frac{4-4x}{2\pi} = \frac{2}{\pi}(1 - x) \), and \( A = \pi \left( \frac{2}{\pi}(1 - x) \right)^2 + x^2 = \frac{4}{\pi}(1 - x)^2 + x^2 \). Then \( A' = \frac{4}{\pi}2(1 - x)(-1) + 2x = \frac{-8 + 8x + 2\pi x}{\pi} \).

Setting this equal to zero, we get the critical point \( x = \frac{4}{4+\pi} \). Now note that the smallest \( x \) can be is zero (when there is only the circle), and the largest it can be is 1 (when there is only the square).

So applying the Extreme Value Theorem on the interval \([0, 1]\), we have:

- \( A(0) = \pi \left( \frac{2}{\pi}(1 - 0) \right)^2 = \frac{4}{\pi} \)
- \( A \left( \frac{4}{4+\pi} \right) = \pi \left( \frac{2}{\pi} \left(1 - \frac{4}{4+\pi} \right) \right)^2 + \left( \frac{4}{4+\pi} \right)^2 = \frac{4}{\pi} \left( \frac{\pi}{4+\pi} \right)^2 + \left( \frac{4}{4+\pi} \right)^2 = \frac{4}{4+\pi} \)
- \( A(1) = 1^2 = 1 \)

Then, we get

(a) a maximum at \( x = 0 \) and \( r = \sqrt{\frac{4}{\pi}} \), ie when we construct only the circle.

(b) a minimum at \( x = \frac{4}{4+\pi} \) and \( r = \frac{2}{4+\pi} \), and so we should cut \( \frac{16}{4+\pi} \) ft of wire to construct the square, and use the rest to make a circle.