To the Chairman of Examiners for Part III Mathematics

Dear Sir,

Find enclosed the Part III essay of Bruce Fontaine.

Signed

(Director Of Studies)
Alternating Knots

Bruce Fontaine

Trinity Hall

I declare that this essay is work done as part of the Part III Examination. It is the result of my own work, and except where stated otherwise, includes nothing which was performed in collaboration. No part of this essay has been submitted for a degree or qualification.

Signed

Address:
298 Huron St.
London, Ontario
Canada, N6A2K2
Alternating Knots
1 Introduction

1.1 Definition. A knot, respectively link is an embedding of $S_1$ in $S^3$, respectively embeddings of disjoint copies of $S^1$ in $S^3$. If $L$ is a link, we take $L$ to denote its image in $S^3$. Two links, $L_1$ and $L_2$, are equivalent if they are ambient-isotopic. That is, there exists a homeomorphism $h : S^3 \to S^3$ with $h(L_1) = L_2$ that is homotopic to the identity on $S^3$.

Note that although this essay is mainly concerned with knots, many results are easier to prove in the general class of links. We will only consider those links that are ambient-isotopic to piecewise linear curves. This is the same as specifying that each embedding of $S^1$ in a link $L$ is smooth [2]. Those that are not smooth embeddings are call wild. As is usual in mathematics, $L$ will be used to denote both the link as a subset of $S^3$ and its equivalence class. A link is called trivial if it is equivalent to multiple disjoint copies of the circle in a plane. The unknot Figure 1(b), is the sole trivial knot. At this point, it is clear that there are non trivial links, see Figure 1(a). On the other hand, it is reasonable to ask if there are knots that are not equivalent to the unknot. By the end of this section, we will have resolved this question.
1.2 Definition. Given a link $L$, consider its perpendicular projection onto a 2-sphere in $S^3$; this is effectively a projection onto $\mathbb{R}^2$, $p : L \to \mathbb{R}^2$. We call such a projection regular if the following conditions are satisfied.

(i) For any $x \in \mathbb{R}^2$, $|p^{-1}(x)| \leq 2$.

(ii) For those points such that $|p^{-1}(x)| = 2$, the crossing is transverse.

(iii) $\{ x : x \in \mathbb{R}^2, |p^{-1}(x)| = 2 \}$ is finite.

If $x$ is a point in the projection and $p^{-1}(x) = \{ y, z \}$, then by (ii), in a small ball around $x$, the projection of the segment of the link passing $y$ divides the ball into two regions, each containing a portion of the projection of the segment of the link passing through $z$. That is, the segments of the link projection actually cross.

1.3 Theorem. Every link $L$ has a regular projection

We will assume this theorem without proof. However, it is fairly easy to imagine that this is valid: if a projection is not regular, a small perturbation will make it so. A regular projection is also known as a link diagram. Since we are essentially working with smooth embeddings, we can choose a point at infinity and the projection then becomes one from $R^3$ to $R^2$ and when drawn in the plane, for each point whose preimage is of size 2, we draw the segment that is closer to the projection plane with a gap to denote that it goes 'under', while the closer segment goes 'over'. Each of these points is called a crossing. Figure 1 (a) shows such a link diagram.

1.4 Definition. A link diagram is said to be an $n$ crossing diagram if there are $n$ crossings. A link is said to be an $n$-crossing link if the diagram with the least number of crossings is an $n$ crossing diagram. Such a diagram is called a minimal diagram for the link.

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1Diagrams thanks to Knotilus: http://knotilus.math.uwo.ca
1.5 Definition. Each embedding of $S^1$ in a link is called a *component* and a link is said to have $k$ components if it consists of $k$ embeddings of $S^1$. If the two arcs of a crossing belong to the same component, we call it a *component crossing*, otherwise it is a *link crossing*.

1.6 Definition. A link is called *alternating* if it admits a diagram such that, when one traverses the link, one alternates between over and under at subsequent crossings.

1.7 Definition. Given an orientation on each component in a link diagram $D$, the *writhe* of a crossing is either $+1$ or $-1$ as given in Figure 2. Summing the writhe for all crossings in $D$ gives $w(D)$ the writhe of the diagram. The writhe of a component crossing is not dependent on the orientation of the component, hence the writhe of a diagram of a knot does not depend on the orientation. For a link, two different orientations of the same diagram can give different writhe: take the Hopf Link, we can have a writhe of $+2$ or a writhe of $-2$ depending on the orientation. As well, the writhe is not constant for all diagrams of a knot.

![Figure 2: The Writhe of a Crossing](image)

1.8 Definition. A *split link* is a link $L$ such that there exists a 2-sphere in $S^3 - L$ that splits $S^3$ into two balls, each containing at least one component of $L$. A link diagram will be called *connected* if it is a connected subset of $\mathbb{R}^2$, clearly a non-split (connected) link admits only connected diagrams. Note that the fact that we draw diagrams with a gap to denote an under crossing does not mean that such a gap actually exists in the diagram. A prime link is a non-split link that is not the sum of two links $L_1$ and $L_2$, that is, it is not ambient-isotopic to the result of the knot produced by joining the diagrams for $K_1$ and $K_2$ as in Figure 3.

1.9 Definition. A link diagram is said to be reduced if every crossing is the meeting point of 4 unique regions.
1.10 Theorem. (Reidemeister) Two link diagrams $D_1$ and $D_2$ give rise to ambient-isotopic links $L_1$ and $L_2$ iff $D_1$ can be transformed into $D_2$ via a finite sequence planar isotopy and of the three Reidemeister moves in Figure 4. Note that the crossing removed by an $\Omega_1$ move is called a nugatory crossing.

Given a crossing, we can consistently label the opposing sides with either +1 or −1 as in Figure 5 a. Thus we can define what it means to smooth the crossing in the +1 or the −1 direction, as seen in Figures 5(b)–(c).

1.11 Definition. The bracket polynomial $\langle D \rangle$ of a link diagram $D$ is defined by the following recursive rules:

(i) $\langle O \rangle = 1$ where $O$ is the standard diagram of the unknot.

(ii) Pick a crossing $x$, then $\langle D \rangle = A \langle D_x^+ \rangle + A^{-1} \langle D_x^- \rangle$ where $D_x^+$ and $D_x^-$ have the crossing $x$ smoothed in the positive and negative directions respectively.

(iii) $\langle O \cup D \rangle = (-A^2 - A^{-2}) \langle D \rangle$

Since we can apply the definitions recursively to a link diagram, a polynomial satisfying these conditions exists. Now, we will demonstrate that the polynomial is uniquely determined by these requirements. By a state $s$ of an $n$-crossing diagram $D$, we mean a choice of smoothing in the +1 or −1 direction for each crossing. More precisely, a state $s$ is a function from $\{1, 2, \ldots, n\} \rightarrow \{1, -1\}$ if $n > 0$ while $s$ is defined to be 0 if $n = 0$. Let $\sum s = \sum_{i=1}^n s$ when $n > 0$ and $\sum s = 0$ when $n = 0$. For a state $s$, smoothing crossing $i$ in the direction $s(i)$ results in some number $\gamma(s)$ of disjoint closed curves in the plane. Now, a complete expansion of the bracket gives a sum over all possible states of terms with the form (here we are using (i) and (ii) to complete the expansion) $A^\sum s (-A^2 - A^{-2})^{\gamma(s)-1}$ so

$$\langle D \rangle = \sum_s A^\sum s (-A^2 - A^{-2})^{\gamma(s)-1}.$$
(a) Reidemeister Type 1 ($\Omega_1$) move

(b) Reidemeister Type 2 ($\Omega_2$) move

(c) Reidemeister Type 3 ($\Omega_3$) move

Figure 4: The Reidemeister Moves

(a) $-1$ smoothing
(b) $+1$ smoothing

Figure 5: Labelling of regions and smoothing of crossings
1.12 Definition. Let $D$ be an oriented link diagram. The Kauffman polynomial is
\[ (-A^3)^{-w(D)} \langle D \rangle. \]
The Jones polynomial, $J(D)$ is the Kauffman polynomial evaluated at $A = t^{-\frac{1}{4}}$.

1.13 Theorem. The bracket polynomial is invariant under $\Omega_2$ and $\Omega_3$ and if $D'$ is one $\Omega_1$ away from $D$, then $\langle D \rangle = A^\pm 3 \langle D' \rangle$.

Proof. First we check invariance under $\Omega_2$:

\[
\langle \overbrace{\begin{array}{c} \circ \bigcirc \end{array}} \rangle = A \langle \begin{array}{c} \bigcirc \circ \end{array} \rangle + A^{-1} \langle \begin{array}{c} \bigcirc \bigcirc \end{array} \rangle \\
= -A^{-2} \langle \begin{array}{c} \bigcirc \circ \end{array} \rangle + \langle \begin{array}{c} \bigcirc \bigcirc \end{array} \rangle + A^{-2} \langle \begin{array}{c} \bigcirc \circ \end{array} \rangle
\]

Then under $\Omega_3$:

\[
\langle \overbrace{\begin{array}{c} \bigcirc \bigcirc \end{array}} \rangle = A \langle \overbrace{\begin{array}{c} \bigcirc \bigcirc \end{array}} \rangle + A^{-1} \langle \overbrace{\begin{array}{c} \bigcirc \bigcirc \end{array}} \rangle \\
= A \langle \overbrace{\begin{array}{c} \bigcirc \bigcirc \end{array}} \rangle + A^{-1} \langle \overbrace{\begin{array}{c} \bigcirc \bigcirc \end{array}} \rangle \\
= \langle \overbrace{\begin{array}{c} \bigcirc \bigcirc \end{array}} \rangle
\]

Finally, for $\Omega_1$ on a crossing of positive writhe:

\[
\langle \overbrace{\begin{array}{c} \bigcirc \end{array}} \rangle = A \langle \begin{array}{c} \bigcirc \end{array} \rangle + A^{-1} \langle \begin{array}{c} \bigcirc \end{array} \rangle \\
= (A(-A^{-2} - A^2) + A^{-1}) \langle \begin{array}{c} \bigcirc \end{array} \rangle \\
= -A^3 \langle \begin{array}{c} \bigcirc \end{array} \rangle
\]

while for $\Omega_1$ on a crossing of negative writhe, $\langle \overbrace{\begin{array}{c} \bigcirc \end{array}} \rangle = -A^{-3} \langle \begin{array}{c} \bigcirc \end{array} \rangle$. \qed

1.14 Corollary. The Jones polynomial is an invariant for links.

Proof. Since both the bracket polynomial and writhe are invariant under $\Omega_2$ and $\Omega_3$, we only need check $\Omega_1$ moves. Suppose that $D'$ is obtained from $D$ by $\Omega_1$ in such a way that we removed a crossing of writhe $v$, then $w(D) = v + w(D')$ and $\langle D \rangle = -A^{3v} \langle D' \rangle$, so

\[
(-A^3)^{-w(D)} \langle D \rangle = (-A^3)^{-v-w(D')}(-A^{3v} \langle D \rangle) = (-A^3)^{-w(D')} \langle D' \rangle.
\]

Thus the Jones polynomial is an invariant of knots. \qed
At this point we are in a position to prove the existence of non-trivial knots. The Jones polynomial for the unknot is 1 and a simple calculation shows that it is \( t + t^3 - t^4 \) and \( t^{-1} + t^{-3} - t^{-4} \) for the two trefoil knots in Figure 6(a)–(b). One can also use the Jones polynomial to establish the existence of non-alternating knots as well. Using results from the next section, knot 8_{19} (Figure 6(c)) is clearly prime and has Jones polynomial \( -t^8 + t^5 + t^3 \).

If 8_{19} is alternating, then any reduced alternating diagram for it would have \( \text{span}(−t^8 + t^5 + t^3) = 5 \) crossings. But the two prime alternating knots of 5 crossings have Jones polynomials different from \( -t^8 + t^5 + t^3 \).

![Figure 6: Some knots](image)

2 Tait Conjectures

The Tait Conjectures concern alternating links and amount to the following:

1. All reduced alternating diagrams for a link have the minimal number of crossings.

2. All reduced alternating diagrams for a knot share the same writhe.

3. All reduced alternating prime diagrams for an alternating link are connected by a finite number of flype moves. This is known as the Tait flyping conjecture.

Tait proposed these conjectures about 130 years ago, and they have only been proved recently. The key to the proofs was the discovery of the Jones polynomial by Vaughn Jones in 1983. The first two conjectures were proved independently by Kauffman, Murasugi and Thistlethwaite in 1987. The third, was proved by Menasco and Thistlethwaite in 1991.
2.1 The First Tait Conjecture

All three proofs of the first conjecture involve the Jones/Kauffman polynomial. The version of the proof that follows is based on that in [6], which itself is loosely based on Murasugi's in [11].

If \( f \) is a single variable Laurent polynomial, \( \text{span}(f) \) shall denote the difference of the highest and lowest powers. Note that by 1.13, for any two diagrams \( D, D' \) of a link \( L \), \( \text{span}(D) = \text{span}(D') \). Our goal is to show that for an alternating link, the span of the Jones polynomial is both the minimal number of crossings and the number of crossings in a reduced alternating diagram, or equivalently, that \( \text{span}(D) = 4n \), where \( n \) is the number of crossings in a reduced alternating diagram of the link.

2.1 Lemma. If \( D \) is an alternating link diagram, the labelling of the regions at each crossing with \( \pm 1 \) is consistent within a face of the underlying graph.

Proof. For a face \( F \) start walking along an edge from the under-crossing, if the interior of the face is to the left we see a \(-1\) and then when we come to the end of the edge (an over-crossing), we once again see a \(-1\) due to the labelling; similarly we would see \(+1\) if the interior is to the right. Thus a face is labelled consistently as in Figure 7.

\[
\begin{array}{c|c}
-1 & -1 \\
- & -1 \\
\end{array}
\]

Figure 7: A typical face in an alternating link diagram

For a diagram \( D \) with \( n \) crossings, let \( s_+ \) and \( s_- \) denote the two constant functions \(+1\) and \(-1\) if \( n > 0 \), while \( s_+ = s_- = 0 \) if \( n = 0 \).

2.2 Lemma. If \( D \) is a connected link diagram with \( n \) crossings, we have:

1. \( \gamma(s_+) + \gamma(s_-) = n + 2 \) if \( D \) is alternating.
2. \( \gamma(s_+) + \gamma(s_-) \leq n + 2 \) otherwise.

Proof of (a). If the diagram is alternating, when we smooth a crossing in the positive direction, we are joining two faces labelled \(+1\) and if we smooth in the negative direction, we are joining two faces labelled \(-1\). Take \( D|_{s_+} \), the application of \( s_+ \) to the diagram \( D \). In this diagram, we have joined all
of the positive regions into a one, hence we have a boundary loop for each face labelled with $-1$. Similarly for $D|s_-$ we have a boundary loop for each face labelled $+1$. Hence $\gamma(s_+) + \gamma(s_-)$ is the number of faces, which Euler’s theorem tells us is $n + 2$.

Proof of (b). We prove the inequality by induction on the number of crossings in the diagram. If $n = 0$, $s_+ = s_- = 0$ and $\gamma(s_+) = 1$. Now, assume we have a diagram with $m$ crossings and that the hypothesis is true for all $n < m$. Let $D$ be a link diagram with $m$ crossings and pick a crossing $x$. Of the two directions that we can smooth $x$, one may result in a split link diagram.

Suppose that smoothing in the $+1$ direction results in a connected link diagram. Call the diagram produced by this smoothing $D'$. It has $m-1$ crossings, thus $\gamma(s'_+) + \gamma(s'_-)$ $\leq m + 1$ and $\gamma(s'_+) = \gamma(s_+)$ while $\gamma(s'_-) = \gamma(s_-) \pm 1$. If smoothing in the $-1$ direction results in a connected diagram, then a similar argument shows that $\gamma(s'_-) = \gamma(s_-)$ and $\gamma(s'_+) = \gamma(s_+) - 1$. In either case, $\gamma(s_+) + \gamma(s_-) \leq m + 2$, as required.

Each state contributes some terms to the Kauffman bracket and we shall show that $s_+$ contributes the largest power while $s_-$ contributes the smallest power. Recall the formulation of the bracket in terms of states,

$$\sum_s A^{\sum s} (-A^2 - A^{-2})^{\gamma(s)-1}.$$ 

So the largest exponent that a state $s$ could contribute is $\sum s + 2\gamma(s) - 2$ and the smallest is $\sum s - 2\gamma(s) + 2$.

2.3 Theorem. Let $D$ be a reduced alternating diagram of a link and $s$ a state of $D$. If $\sum s = n - 2$ then $\gamma(s_+) > \gamma(s)$, while if $\sum s = 2 - n$ then $\gamma(s_-) < \gamma(s)$.

Proof. Since $D$ is a reduced alternating diagram, each crossing is a vertex in the boundary of 4 unique regions. If $\sum s = n - 2$, then $s$ is obtained from $s_+$ by switching one crossing from a $+1$ smoothing to a $-1$ smoothing. Now, the $-1$ regions adjacent to the crossing are not the same since the diagram is reduced alternating, hence $\gamma(s) < \gamma(s_+)$ since we are amalgamating two disjoint regions into one, joining the two boundary curves into one. The proof is similar for $s_-$.

2.4 Lemma. Suppose $s$ is a state of the link diagram $D$, then

$$\sum s + 2\gamma(s) - 2 \leq \sum s_+ + 2\gamma(s_+) - 2$$
and
\[ \sum s_- - 2\gamma(s_-) + 2 \leq \sum s - 2\gamma(s) + 2, \]
with strict inequalities when \( D \) is reduced alternating.

**Proof.** First, select a state \( s' \) derived from \( s \) by switching all but one \( -1 \) smoothing to \( +1 \), then \( \sum s' = n - 2 \). Suppose that \( s \) has \( k \) negative smoothings. Each time we change the direction of a smoothing to obtain \( s' \), the number of loops changes by \( \pm 1 \). So, if \( s \) has \( k \) negative smoothings, we have
\[ \gamma(s) \leq \gamma(s') + (k - 1), \]
so
\[ \sum s + 2\gamma(s) - 2 \leq \sum s' - 2(k - 1) + 2(\gamma(s') + (k - 1)) - 2 = \sum s' + 2\gamma(s') - 2. \]

From the previous theorem, when \( D \) is reduced alternating, we then see that
\[ \sum s' + 2\gamma(s') - 2 = \sum s_+ - 2 + 2\gamma(s') - 2 < \sum s_+ + 2\gamma(s_+) - 2 \]
as needed. Otherwise, we have
\[ \sum s' + 2\gamma(s') \leq \sum s_+ - 2 + 2(\gamma(s_+) + 1) - 2 = \sum s_+ + 2\gamma(s_+) - 2. \]
The proof of the second equality is similar.

**2.5 Theorem.** If \( D \) is a connected link diagram with \( n \) crossings, \( \text{span}(D) \leq 4n \) with equality when \( D \) is reduced alternating.

**Proof.** By the previous theorems, we have shown that the maximum and minimum possible exponents result from states \( s_+ \) and \( s_- \) respectively. The strict inequality assertions of Lemma 2.4 shows that they are guaranteed to be present if the diagram is reduced alternating, hence:
\[ \text{span}(D) \leq \sum s_+ + 2\gamma(s_+) - 2 - \left( \sum s_- - 2\gamma(s_-) + 2 \right) \]
\[ = n + 2\gamma(s_+) - 2 - (-n) + 2\gamma(s_-) - 2 \]
\[ = 2n + 2(\gamma(s_+) + \gamma(s_-)) - 4 \]
\[ = 2n + 2(n + 2) - 4 \]
\[ = 4n \]
where the first inequality can be replaced by equality when \( D \) is reduced alternating.

**2.6 Corollary.** If \( D \) is a reduced alternating diagram for a link \( L \), then it is a minimal diagram for \( L \).

**Proof.** Let \( L \) be a link, and let \( D, D' \) be \( n \), respectively \( m \) crossing diagrams for \( L \), with \( D \) reduced alternating. Then \( 4n = \text{span}(D) = \text{span}(D') \leq 4m \) and so \( n \leq m \).
2.2 The Third Tait Conjecture

In 1991, Menasco and Thistlethwaite announced that they had proved the third Tait conjecture [9]. Published in 1993, [10] relies on previous results on link complements and on results due to the Jones polynomial. To understand the theorem, we first start with an informal definition of a flype. We note that a tangle is a region of a link diagram enclosed by a simple closed curve, said to be an \( n \)-tangle if there are \( n \) intersections with the link. Due to a result of Menasco [8], an alternating link diagram is prime if and only if it admits no trivial 2-tangles. Now a flype move on a link diagram involves locating a 4-tangle with an adjacent crossing and twisting the tangle a half turn in the appropriate direction, destroying the crossing on one side and creating it on the other as seen in Figure 8.

![Figure 8: A typical flype move](image)

To formalize this notion, for a diagram \( D \) of a link \( L \) of \( n \) crossings, let \( B_1, B_2, \ldots, B_n \) be small ‘crossing ball’ neighbourhoods of each crossing. Without loss of generality, we assume that \( L \) is taken to be \( D \), except in the crossing balls where the crossings are perturbed outwards; we say \( L = \lambda(D) \). We assume that we are working in \( S^3 \) as \( \mathbb{R} \cup \{ \infty \} \), that \( D \) is embedded in \( S^2 = \{ x : ||x|| = 1 \} \) and that \( L \) lies in \( S^2 \times [1/2, 3/2] \). We define a standard pattern to be as in Figure 8(a). The two tangles \( A \) and \( B \) lie inside the euclidean 2-spheres \( S_A \) and \( S_B \). The 2-sphere \( B'_x \) meets \( L \) in two parallel segments.

2.7 Definition. Given a diagram \( D_1 \) with a standard pattern, a standard flype for this pattern is any homeomorphism \( f : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2)) \) where \( D_2 \) corresponds to Figure 8(b). That is:

(i) \( f \) maps \( S_A \) to \( S_A \) via a rigid rotation through \( \pi \) about an axis parallel to the projection plane.
(ii) $f$ fixes pointwise $S_B$.

(iii) $f$ maps $B_x$ and $B'_x$ to themselves via a half twist.

If $f_1$ and $f_2$ are two standard flypes of a pair $(S^3, \lambda(D))$ for the same standard pattern, $f_2^{-1} \circ f_1$ is pairwise isotopic to a homeomorphism of $S^3$ which fixes $\lambda(D)$. Hence it is clear that $f_2^{-1} \circ f_1$ is isotopic to the identity and $f_1$ and $f_2$ are pairwise isotopic. Now, this standard flype is too rigid to be of much use, since it requires that the link diagram be in some standard pattern. We need to extend to its equivalence class under orientation preserving homeomorphisms of $S^2$. In this vein, we need to define a class of homeomorphisms of $S^3$ that are trivial with respect to $S^2$. Let $N = \{ x : 1/2 \leq ||x|| \leq 3/2 \}$, that is $N \cong S^2 \times [1/2, 3/2]$.

2.8 Definition. A homeomorphism, $g : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$ is flat if it is pairwise isotopic to a homeomorphism $h$ such that $h(N) = N$ and $h|_N = h_0 \times id_{[1/2, 3/2]}$ where $h_0$ is some orientation preserving homeomorphism of $S^2$.

2.9 Definition. A flype is a homeomorphism $f : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$ that can be written $f = g_1 \circ f' \circ g_2$ where $g_1$, $g_2$ are flat and $f'$ is a standard flype.

Given these definitions, Menasco and Thistlethwaite proved the following theorem:

2.10 Theorem. If $D_1$ and $D_2$ are two reduced, prime, oriented, alternating link diagrams and if $f$ is an orientation preserving homeomorphism of pairs $f : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$, then $f$ is a composite of flypes.

In order to prove the theorem, Menasco and Thistlethwaite use induction on the number of crossings of 4-regular rigid vertex graphs (we will use the word graph to mean this from now on in this section). By a rigid vertex, we mean that we can consider the vertex to be small flat 2-dimensional disc which is mapped by the morphisms of the category to a flat 2-dimensional disc, i.e. the disc is not twisted or bent by the morphism. Similar to link diagrams, we can consider diagrams of these graphs, with over crossings, under crossings and vertices. So, if $D$ is a diagram of a link $L$, then it is also the diagram of a graph with no vertices. This makes the category of 4-regular rigid vertex graphs seem larger thereby making the proof more general, but we shall show that this is in fact not the case later in this section. We can define the same properties that we have been requesting of knots for these graphs as well. We say a diagram $D$ of a graph is prime alternating and
reduced if we can replace each vertex by an over and under crossing pair such that the resulting diagram is alternating, prime and reduced. An orientation of the graph is simply an orientation of each edge.

As well, the idea of a flype as outlined for links naturally extends to graphs, with the case of a graph with no crossings being trivial, and hence the base case for the induction. Menasco and Thistlethwaite give 3 separate induction arguments, called Inductive Arguments A, B and C, at least one of which applies to each link.

1. Inductive Argument A: If we are given reduced, alternating, prime, oriented diagrams $D_1$ and $D_2$ along with a homeomorphism of pairs $f : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$, it is first shown that $D_1$ and $D_2$ have the same number of crossings. Assuming this is at least 1, since we are proving the inductive step, the goal is to show that $f$ is pairwise isotopic to some map $f'$ which maps a crossing ball of $D_1$ to a crossing ball of $D_2$ isometrically. If this is the case, we replace the crossing by a rigid-vertex, so $f'$ induces a morphism of graphs and is, by the inductive hypothesis, a compositions of flypes. The inductive argument shows that if certain geometric properties hold, then there are flypes $\phi_1 : (S^3, \lambda(D_1')) \to (S^3, \lambda(D_1))$ and $\phi_2 : (S^3, \lambda(D_2)) \to (S^3, \lambda(D_2'))$ such that $\phi_2 \circ f' \circ \phi_1$ maps some crossing ball of $D_1'$ isometrically to some crossing ball of $D_2'$.

2. Inductive Argument B: We assume $D_1$, $D_2$ and $f$ are defined as above. If it is possible to take a simple closed curve on $S^2$ that intersects transversely some vertex $v_2$ and some crossing $x_2$ of $D_2$, then it is shown that a similar curve exists for $v_1 = f^{-1}(v_2)$ and some crossing $x_1$. We can then flype $x_1$ and $x_2$ to the same side of $v_1$ and $v_2$ according to $f$, and the resulting composition is as above. It will map the crossing ball of $x_1$ under the flype to the crossing ball of $x_2$ under the flypes. Hence the inductive argument holds.

3. Inductive Argument C: We assume $D_1$, $D_2$ and $f$ are defined as above. Let $\Sigma_1$ be a 2-sphere whose intersection with $S^2$ is a circle separating $D_1$ into two 4-tangles. Suppose that each one has at least 1 crossing, and that we can show that, up to pairwise isotopy, $f(\Sigma_1) = \Sigma_2$, having the same properties as $\Sigma_1$ but with respect to $D_2$. Then for each sphere $\Sigma_1$ and $\Sigma_2$ respectively, replace the part of the graph inside the sphere by a single rigid vertex. We can do the same for the complementary 3-balls. To each of these situations, we apply the inductive hypothesis and then paste the result together along the boundary of the rigid 3-balls, thereby giving the desired result.
As was mentioned earlier, it seems that this argument is more general than the theorem requires, but we note that each graph with \( c \) crossings and \( v \) vertices can be made into a link with \( c + 5v \) crossings by inserting a crossing surrounded by a loop at each vertex. This maintains the rigidity of the vertex, since a homeomorphism of pairs involving this diagram, cannot disturb the layout of the crossings that we have inserted.

2.3 The Second Tait Conjecture

By the third conjecture, all reduced alternating diagrams for a knot are connected via flype moves. When doing a tangle turn, the writhe of the tangle we are turning does not change. As well, for the part of the knot that is unaffected by the flype move, the writhe remains the same. Since the crossing that was created by the flype has the same writhe as the crossings that was destroyed, it follows that all reduced alternating diagrams for a given knot have the same writhe.

3 Enumeration

3.1 Introduction

Prior to the proof of the Tait flyping conjecture, enumeration inevitably involved a human component. Even with the advent of the computer, the naive method of generating all possible alternating link diagrams and then grouping them into flype equivalence classes would still have required some outside check to see that different classes represented different links. With the conjecture proven, this method can be carried out solely by computer.

First, we need a method of representing a link in a computer. One of the simplest to describe, but by no means the most compact, is the Gauss code. To write a Gauss code for a link diagram of \( n \) crossings, we label each crossing with a unique integer from 1 to \( n \). Then, we traverse each component, writing the labels down in the order we see them with a + or – when we see them as an over or under crossing. We separate the labels for the same component with commas and separate the components with colons. For an alternating link diagram, the sequence of + and – alternates, so we can ignore them (up to mirror image). The result is a sequence of the integers from 1 to \( n \), each occurring twice, with a comma or a colon between successive entries.

We start the algorithm by generating all such sequences. Because not all such sequences encode links (see Kauffman’s work on virtual links), we
then need to weed out the invalid codes. Next, we group the codes that represent the same diagram, discard the non-reduced diagrams and the non-prime diagrams, then use flyping to group them into complete collections of alternating diagrams for links. The major problem with this approach is that there are many more invalid codes than valid ones. In fact, the total number of codes is of factorial order while the number of valid codes is asymptotically exponential. It is therefore evident that the process of generating all sequences only to discard the invalid ones is computationally inefficient.

Rankin, Flint and Schermann developed a much more efficient scheme, using a complete collection of knots at \( n - 1 \) crossings to generate those at \( n \) crossings. We will outline a method of generating prime alternating links based on their methods described in [12] and [13].

Before continuing, we need to decide what we should enumerate. First, we should only look at prime links, since all composite links can be built from them. We should also ignore orientation and the fact that a link and its mirror image (switching the over/under behaviour at each crossing) may not be the same. Once we have a complete collection of prime alternating links up to mirror and orientation, there exist algorithms to decide the number of orientations a link has and if it is equal to its mirror image. From this point on, we will only consider prime alternating links up to mirror image.

3.1 Definition. If we consider a reduced alternating link diagram as a graph, a group is a maximal sequence of crossings \( c_1, \ldots, c_k \) such that \( c_i \) is connected via two edges to \( c_{i+1} \) for \( 1 \leq i < k \). If both arcs of \( c_1 \) are from the same component we shall call it a component group, otherwise it is a link group. Given an orientation of a link, for each group of size 2 or greater we can infer a sign, + if the edges connecting the subsequent crossings are oriented in the same direction or − if they have opposite direction. For component groups, this sign does not depend on the orientation but it does for link groups. We can label each group with a unique label, \( \pm k \); were the group is the \( i \)-th group of size \( k \) along with its sign if \( k > 1 \). Then using the same method as the Gauss code, we can write a group code. We call a group of size 1 a loner, and any loner that flypes can also be given a sign. If we were to replace a flyping loner with a positive or negative group, one would flype and the other would not, so we can use this sign. For example a group code for the trefoil knot is \( +3_1, +3_1 \) and for the Hopf link \( +2_1 : +2_1 \).

3.2 Definition. Given a group of size \( k \), when we turn an adjacent 4-tangle by \( k \) half turns in the appropriate direction, we effect a flype move on the group. Given a group \( k_i \), we lay the link out as in Figure 9 showing all possible flype positions. Those 4-tangles that contain at least one crossing
and are not groups that can flype, are called the \textit{min-tangles} for the orbit of \(k_i\) and the \textit{flype orbit} is the set of min-tangles. The remaining non trivial 4-tangles are thus groups that can flype and will share the same flype orbit as \(k_i\). We say that those groups that share the same flype orbit are part of the same \textit{full group}. Note that two groups sharing even one min-tangle will have the same flype orbit. Thus it can be shown that we can always flype and amalgamate all groups that share an orbit together into the full group for that orbit. If we do this for all groups, we obtain a full group diagram for the link, thus each link has at least 1 full group diagram.

![Figure 9: Flype Orbit](image)

### 3.2 The Master Array

Given a group code for a link, we will construct a \textit{master group code}, containing all the flyping information of all groups in the link. To do this we will use the following theorem from [12] without proof:

**3.3 Theorem.** For a full group diagram, given two groups \(G_1\) and \(G_2\), all flype positions of \(G_1\) are contained in a min-tangle for \(G_2\) and all flype positions of \(G_2\) are contained within a min-tangle for \(G_1\).

This says that it does not matter in what order we flype groups to achieve a specific full group configuration nor does it matter in what order we calculate the flype positions of the groups. To construct a master group code for a given orientation of a link, we take a reduced alternating full group diagram \(D\) and for each group \(k_i\) we carry out the following process: we think of the diagram in the form of Figure 9, we locate its min-tangles and drop successive labels \(k_j^i\) label on the parallel arcs between consecutive pairs of min-tangles. Once we have done this, we again follow the same process used to generate the group code, noting down each label as we pass it. The key of the master group code is that up to a permutation of the super and subscripts, all master group codes for a given oriented link are the same. Reversing the orientation of a component in the link results in the reversal of the code for that component in the master group code and the reversal of signs on all
link groups involving that component, so changing the orientation results in a well defined change in the code.

Some examples: in the simple case of a link with no flyping groups, say the figure eight knot, the group code and master group code are essentially the same:

\[-2_1, -2_2, -2_1, -2_2\]

to

\[-2_1^0, -2_2^0, -2_1^0, -2_2^0\].

For the following link code

\[2_1, -1_1, -2_2, -1_1 : 2_1, -2_2\]

a master group would be

\[2_1^0, -1_1^0, -2_2^0, -1_1^0 : 2_1^0, -1_1^1, -2_2^0, -1_1^1\].

Now that we know how to construct a master group code for a link, we introduce a function that takes as input any master group code for a link and returns a uniquely determined master group code, called the master array for the link (see [3] for details). Information that is generated by this function also provides us with the ability to determine for a given prime alternating link a representation of its symmetry group, the number of orientations, and whether the link is chiral or invertible. This is discussed in the upcoming papers of Rankin, Flint, and Fontaine on link enumeration. It should also be noted that once we have the master array, dropping all but the first occurring flype position of each group gives us a unique group code representing the knot, the zero position group code and hence a zero position gauss code, from which we can easily recover the master array by simply adding the flype positions back in. The master array is thus a complete invariant for prime alternating links and can be extended to composite links via choosing some ordering on the prime summand of a composite and producing a master array for each.

3.3 Enumeration Scheme

The master array allows us to directly enumerate the prime alternating links with \(n\) crossings and \(k\) components via a 2 stage process.

3.3.1 Stage 1

3.4 Definition. The main operation that is performed is called doubling a crossing: given an oriented prime alternating link diagram, it replaces a
single crossing by a 2-group. For a given crossing, there are two possible ways to perform a doubling operation, and we call these the directions of doubling; both directions result in a prime alternating link diagram. While one of the direction at each crossing will maintain the number of components, the other will change that number; the number of components will be increasing by one if the initial crossing was a component crossing and will be decreasing by one if it was a link crossing. Thus, each direction will produce a different link.

(a) Crossing 
(b) Doubling

Figure 10: Doubling a crossing

Once a group and a doubling direction have been selected, it does not matter to what crossing in the group the doubling operation is applied: it will result in the same link. It is easy to see if the doubling operation is done in a direction parallel to the group since it increases the group’s size by 1. On the other hand, if we apply the doubling operation in the opposite direction, we may split the group into two parts, but the resulting 2-group is now a min-tangle in the orbit of the original group; therefore we can flype the crossings from one side to the other amalgamating the split group. With this, the doubling operation is well defined on master group codes up to the orientation of the resulting components. Thus, the result of applying the doubling operation to a group is therefore well defined for master arrays; there are many other details to this that are discussed in [3] and [12].

Given the complete collection of master group codes for the \( n - 1 \) crossing \( k - 1, k \) and \( k + 1 \) component links, we wish to produce the complete collection of all links at \( n \) crossings, \( k \) components that have at least one group of at two or more crossings. To do this, we must apply the crossing doubling operation to certain groups (selected according to criteria that ensure that all the required links will be generated) in each flype position. The direction we double in depends on the number of components of the input link. If it has \( k - 1 \) components, then we need double each component group in the positive direction, producing a new component. If it has \( k \) components we double each component group in the negative direction, leaving the number of components unchanged. Finally if we have \( k + 1 \) components, we double each link group in both directions, losing a component.

3.5 Theorem. The collection of master groups codes formed above contains
at least one code for each \( n \) crossing, \( k \) component prime alternating link that has a group of at least size 2.

Proof. If we have a prime alternating link \( L \) with a group of at least size 2, we can smooth a crossing from that group to obtain a prime alternating link \( L' \) of \( n - 1 \) crossings and \( k - 1, k \) or \( k + 1 \) components. Call the resultant group \( x \) and note its flype position in the diagram \( L' \). Then, when we process \( L' \), we will apply the crossing doubling operation to the group \( x \), thereby reconstructing \( L \).

Now we convert the master groups codes to master arrays and remove duplicates.

3.3.2 Stage 2

Now, given the collection of master group codes above, we iteratively apply the operation shown in Figure 11 which is known as the OTS operation. To perform an OTS operation, we take a face of degree 3 in a diagram and move one edge past the opposite crossing and then fix the crossings so that the link remains alternating. It is clear that it does not matter which edge we choose, the result is the same. We perform all possible operations of this type and then take the collection of newly generated links, and apply the same process. We do this until no new links are generated. To see that this terminates and generates the rest of the prime alternating links, we need the following theorems with outlines of the proofs given:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ots_operation.png}
\caption{The OTS Operation}
\end{figure}

3.6 Definition. A plane graph \( G \) is called a 2-region, respectively a loop, if

(i) \( G \) is connected.

(ii) \( G \) has a face \( F \) that is bounded by a cycle \( C \).

(iii) There are exactly 2 vertices of degree 2 on \( C \) (called the base vertices of the 2-region) respectively a single vertex of degree 2 (called the base vertex of the minimal loop).
(iv) All other vertices on $C$ have degree 3.

(v) All vertices inside $\mathbb{R}^2 - F$ have degree 4. This is called the interior of the loop or region.

More generally, if $G'$ is a subgraph of a plane graph, with vertex set $V' \subset V$, $G'$ is said to be a 2-region, respectively loop, of $G$ if it is a 2-region, respectively loop, and every vertex in the interior of $G'$ belongs to $V'$ and $G'$ is the induced graph on these vertices. We should note that there is no requirement that $F$ be the unbounded or bounded region. A 2-region $G$ is said to be minimal if $H$ is a 2-region of $G$ implies $H = G$ and it contains no loops. A loop $G$ is minimal if $H$ is a loop of $G$ implies $H = G$. For a 2-region, we shall refer to the 2 segments of the cycle $C$ that are bounded by the base vertices as the sides of the 2-region. By the word strand, we mean a path connected subset of one of the components of the link.

3.7 Theorem. Every prime link diagram contains some 2-region.

Proof. Suppose that we have at least 1 component crossing, then starting at this crossing we traverse the component of the link until it first intersects itself, call this vertex $v$. Since we have no nugatory crossings, the partition of the link contained inside the strand from $v$ to $v$ will bound a loop in the diagram. Now, each strand that enters the region must exit, if such a strand crosses itself before it exists, we run the same process again to obtain a loop contained in the previous one. Continue this process until it is no longer possible. Since the number of crossings in the original region was finite, and we remove the base crossing with each iteration, this terminates. The result of this operation will be a minimal loop and an arc entering this loop must exit again without crossing itself, otherwise we violate the construction above. This arc and the portion of the minimal loop it bounds (that not containing the base vertex), is a 2-region.

If we have no component crossings, then select a link crossing, choose one of the components and follow it until it meets the other component again. This must happen because each pair of components must meet in at least 2 crossings. Now, this strand of the first component will form one side of our 2-region, as well it divides the other component into two strands, we select one, and take the region bounded by these two as a 2-region.

3.8 Corollary. Any prime link contains a minimal 2-region.

3.9 Theorem. A minimal 2-region of a prime alternating link can be emptied by a finite number of OTS operations to produce a 2-group.

20
Proof. We proceed by induction on the number of crossings in the interior of the 2-region. To establish the base case of no crossings, we induct on the number of crossings on the boundary. If there are no crossings on the boundary, then the two base crossings are joined by the boundary edges, forming a 2-group. If there are crossings on the boundary, choose the one closest to the base crossing \( x \), call it \( y_1 \). Since there are no crossings in the interior of the region, we follow the non boundary strand through \( y_1 \) into the interior and the next crossing is on the other edge of the minimal 2-region, \( y_2 \). We perform OTS on these 3 crossings and have reduced the number of boundary crossings by 2.

It remains to establish that if a min 2-region contains crossings in its interior, then it can be converted by OTS alone into a min 2-region that contains no crossings in its interior. To prove the inductive step, we must show that for any minimal 2-region, there is at least one 3 face containing exactly one boundary edge. Since the 2-region is minimal, it contains no self intersecting strands and no pair of strands in the region that intersect twice. Thus the \( j \) strands that start at one boundary edge form a braid inside the region and connects to the other boundary edge. If we think of the braid as a sequence of transpositions of adjacent pairs there must be a first one. This pair of adjacent strands that cross form a 3 face with the side of the region and we can apply OTS to remove this crossing. Now, apply the inductive hypothesis.

Thus by performing a finite number of OTS operations, we can make a link that contains no groups into one that does. Thus, by iteratively applying all possible OTS operations to the previous collection, we get a complete collection of links at \( n \) crossings and \( k \) components.

### 3.4 Conclusion

Although the algorithm as described above is far more efficient then the naive approach, with a little more work more computational efficiency can be gained. We can bypass the check for duplicate links after the first stage by following this simple rule: only output the master array of a link if the resultant group after the doubling operation is the first group of largest number of crossings in the master array. We have thus, up to symmetry of the original knot, specified a unique operation used to generate each link. We can also tell, before we apply the operator, if it has a chance of being the first largest group in the master array; if it will never be, i.e. the resultant group would be to small, then we can avoid applying the operation.
The same considerations can be applied to the first iteration of Stage 2, vastly decreasing the number OTS operations applied to generate the second generation. We note that experimental results have shown that 99.999% of the links by the end of the first iteration of Stage 2 are generated by this method and thus we are able to avoid an expensive check to see if we have already generated them. This results in the memory requirements of enumeration up to 23 crossings to be no more then 20KB, so this algorithm can run on a simple desktop computer. As well, the lack of a check against some database means that the first stage and the first iteration of the second stage can be parallelized and hence one can even generate beyond the 23 crossings. We also should note that all the links up to the 23 crossings take 600 GB to store in a compressed state. Experimentally, we have roughly a five to six fold increase with each increment in crossing size, so while the 24 crossing links are within the realm of enumeration, the 25 crossings would require approximately fifteen terabytes of storage space, which is not economically feasible at the present time. On the other hand, if one simply wishes to count the number of links rather then storing them, this is feasible right now.

4 Asymptotic Estimates of the Number of Prime Alternating Links

In [14] it is established that if $A_n$ is the number of prime alternating links at $n$ crossings, then

$$\lim_{n \to \infty} \left( A_n \right)^{\frac{1}{n}} = \frac{101 + \sqrt{21001}}{40}. \quad (2)$$

So the number of prime alternating links grows exponentially. Since we can inject the set of prime alternating knots into the set of prime alternating links of one higher crossing using stage 1 of the enumeration scheme, we thus have an upper bound on the growth rate of the prime alternating knots. The method of Sundberg and Thistlethwaite was two fold. The first result establishes the growth rate of $a_n$, the number of strong equivalence classes of prime alternating tangle types. The second result then relates $a_n$ to $A_n$, the number of prime alternating links.

4.1 Definition. Here we define a tangle as a pair, $(B, T)$ where $B$ is the standard unit ball centered at the origin and $T$ is a proper, tame sub 1-manifold of $B$ such that it meets $\partial B$ at the four points $(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0)$. Two tangles $(B, T_1)$ and $(B, T_2)$ are strongly equivalent if there exists an orienta-
tion preserving homeomorphism of pairs $h : (B, T_1) \rightarrow (B, T_2)$ such that $h$ is the identity on $\partial B$.

They then consider the generating function for $a_n$, $w(z) = \sum a_n z^n$ with $z$ complex and derive the following functional equation by looking at how smaller tangles are used to build larger ones:

$$w(z) = \alpha(z, q(w(z))),$$

where $\alpha$ is derived in their paper and is:

$$\alpha(z, \zeta) = \frac{1}{2} \left( (1 + z - \zeta) - \sqrt{(1 - z + \zeta)^2 - \frac{8}{1 - z} (z^2 - z\zeta + \zeta)} \right)$$

and $q(z)$ is derived from the work of Tutte on generating functions for the number of rooted c-nets and is:

$$q(z) = \frac{1}{2(z + 2)^3} \left( (1 - 4z)^{\frac{3}{2}} + (2z^2 - 10z - 1) \right) - \frac{2}{1 + z} - z + 2.$$

They then use techniques of complex analysis to derive a method for computing $a_n$ explicitly and find that

$$a_n \sim \frac{3c_1}{4\sqrt{\pi}} n^{-\frac{3}{2}} \lambda^{n-\frac{3}{2}}$$

where

$$c_1 = \frac{5^{\frac{3}{2}}}{3^3\sqrt{2} \sqrt{21001 + 371\sqrt{21001}}^3} \sqrt{(17 + 3\sqrt{21001})^3}$$

and

$$\lambda = \frac{101 + \sqrt{21001}}{40}.$$

To calculate the number of prime alternating links given the number of prime alternating tangles, they show that one can select any crossing in an $n$ crossing prime alternating link and the rest of the crossings form a prime alternating tangle. Also, adding a crossing to a prime alternating tangle, connecting the opposite pairs of ends together results in a prime alternating link. Of course, more then one tangle can give rise to the same link, but they calculate that at most $8(2n - 3)$ tangles of $n - 1$ crossings will give rise to the same $n$ crossing link and so the following holds:

$$\frac{a_{n-1}}{8(2n - 3)} \leq A_n \leq \frac{a_{n-1}}{2}$$

from which equation (2) follows.
References


