1. Last assignment we defined the commutator of two elements \([x, y] = xyz^{-1}y^{-1}\). You then showed that the commutator subgroup \([G, G] = \langle [x, y] \mid x, y \in G \rangle\) (generated by all commutators in \(G\)), was a normal subgroup. Show that the group \(G/[G, G]\) is abelian. Moreover, show that \(G/M\) is abelian if and only if \([G, G] \subseteq M\). Thus \(G/[G, G]\) is the ‘largest’ abelian quotient of \(G\).

2. Suppose that \(G\) is a finitely generated abelian group, that is \(G = \langle a_1, \ldots, a_n \rangle\) for some finite set of \(a_i \in G\). Show that any subgroup of \(G\) is also finitely generated. Hint: try to induct on the number of generators. i.e. for \(n = 1\) it should be clear.

3. Show that if \(G/Z(G)\) is cyclic, then \(G\) is abelian.

4. Suppose that \(H\) and \(K\) are normal subgroups of \(G\) with \(H \cap K = \{1\}\), show that \(xy = yx\) for all \(x \in H\) and \(y \in K\).

5. Let \(M\) and \(N\) be normal subgroups of \(G\) such that \(G = MN\). Show that \(G/(M \cap N) \cong (G/M) \times (G/N)\).

6. If \(p\) is a prime and \(G\) is a group of order \(p^2\) show that either \(G\) is cyclic or \(G \cong \mathbb{Z}_p \times \mathbb{Z}_p\).

7. Recall that an action of a group \(G\) on a set \(X\) is called transitive if for all \(x, y \in X\), there exists a \(g \in G\) with \(gx = y\). An action is doubly transitive if \(G_x\) acts transitively on \(X - \{x\}\). Show that \(S_n\) acts doubly transitively on \(\{1, \ldots, n\}\) via the permutation action for \(n \geq 2\).

8. Find the conjugacy classes of the following groups \(\mathbb{Z}_2 \times S_3\) and \(\mathbb{Z}_3 \times A_4\).

9. How many necklaces can be made with six beads of three different colors? The symmetry group should be the dihedral group on 6 elements.

10. Count the number of ways to color the faces of a cube with \(r\) colors (the answer should be a polynomial in \(r\)). The symmetry group here is just made of rotations, since flips are not physical motions.