1. Check if the following sets and group laws satisfy the axioms of a group. If so justify, if not, explain why.

(a) Although this does have an identity, 0, there are no inverses, since \( \max\{a, b\} = 0 \) requires both \( a = b = 0 \).

(b) Note that \( a^2 = 1 \) for any \( a \), and \( a^3 = a \) are inverses, since \( a^2 * a = 1/a \), which are not equal for \( a = 2 \).

(c) For an identity, we need \( e * 1 = 1 \) but the \( 0 \leq a * b < 1 \) for all \( a, b \).

(d) This is a group, \((A, 1_B)\) is the identity, since \((1_A, 1_B)(c, d) = (1_A c, 1_B d) = (c, d)\) for \( c \in A \) and \( d \in B \). Inverses exist: \((c, d)(c^{-1}, d^{-1}) = (1_A, 1_B)\). It is also associative, \((\langle a, b \rangle(c, d))(e, f) = (ac, bd)(e, f) = (ace, bdf) = (a, b)(cd, df) = (a, b)((c, d)(e, f))\).

2. As a corollary to Lagrange’s theorem we have seen \( \text{ord}(a)||G| \) so \( \text{ord}(a) \leq \infty \).

3. Let \( a, b \in G \), then we want to show \( ab = ba \). Starting with \((ab)^2 = 1\), we have \( abab = 1 \). Since \( a^2 = 1 \), multiply on the left by \( a \) to get \( bab = a \) and then on the left again by \( b \) get \( ab = ba \).

4. The easy way to do this is the following: Recall that \( D_6 \) is the symmetry group of the triangle. Label the triangle vertices with the numbers 1, 2, 3, then a symmetry of the triangle gives an element of \( S_3 \) by watching where the vertices go. This gives a map \( f : D_6 \rightarrow S_3 \) which is naturally a group homomorphism, since the composition of transformations results in the composition of the two permutations. Its kernel is trivial since only one transformation, the identity fixes all the vertices. Thus \( f \) is injective and since \( |D_6| = |S_3| \) it is surjective as well.

5. \( f(ab) = (ab)^k = a^k b^k = f(a) f(b) \). The middle step is possible since \( G \) is abelian and \( ab = ba \).

6. Consider the pairing \((a, a^{-1})\) on \( G \). For those elements that are not order 2, \( a \neq a^{-1} \) and thus the total number of element with order higher than 2 is even since they pair up. We also have the element 1, so there are an odd number of elements of order not 2, since \( |G| \) is even, this leaves an odd number of elements of order 2/

7. Note that \( f \) is called a fractional linear transformation. \( f(z) = \frac{az + b}{cz + d} \) and \( h(z) = \frac{ez + f}{gz + h} \). Then

\[
f(h(z)) = \frac{\frac{az + f}{cz + f} + b}{\frac{cz + f}{cz + f} + d} = \frac{a(cez + f) + b(gz + h)}{c(ez + f) + d(gz + h)} = \frac{(ae + bg)z + (af + bh)}{(ce + dg)z + (cf + dh)}
\]

Thus a composition of fractional linear transformation is fractional linear. We need to check the 3 axioms of a group: \( 1_{FL}(z) = z \), the identity function is the identity element under composition. Composition is automatically associative. Finally, inverses: given \( f \) as above, let \( f^{-1}(z) = \frac{dz - b}{ad - bc} \).

One can check that \( f(f^{-1}(z)) = z \), it is obvious from the argument below since \( g(f^{-1}) = (g(f))^{-1} \).

To verify that \( g \) is a group homomorphism,

\[
g(f \circ h) = \begin{bmatrix} a + bg & af + bh \\ ce + dh & cf + dh \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = g(f)g(h).
\]

Note that this assumes that assumes that the map \( g \) is well defined. In fact, if we multiply the numerator and denominator by a constant, the map is not changed, but the coefficients are. So if \( M(2, \mathbb{C}) \) is the set of invertible matrices, in order to make the map well defined it would suffice to replace \( M(2, \mathbb{C}) \) with a quotient \( M(2, \mathbb{C})/H \) where \( H \) is the subgroup formed by complex multiples of the identity matrix. Note that \( H \) is normal. This quotient is often denoted \( PGL(2, \mathbb{C}) \).

8. Suppose that \( h, k \in H \), then there exists some \( n \) and \( m \) such that \( h^n = 1 \) and \( k^m = 1 \). Take \((hk)^{mn} = h^{mn} k^{mn} = 1^{m} 1^{n} = 1 \), thus \( hk \) is finite order. Note that the order of \( hk \) is the least common multiple of \( m \) and \( n \).
9. Since $H$ is generated by the commutators, it suffices to show that $g[a, b]g^{-1} \in H$ for all $a, b, g \in G$. This is because an arbitrary element of the group $H$ is a product $[a_1, b_1][a_2, b_2]...[a_k, b_k]$ of commutators and if $g[a_i, b_i]g^{-1} \in H$, then $g[a_1, b_1][a_2, b_2]...[a_k, b_k]g^{-1} = g[a_1, b_1]g^{-1}g[a_2, b_2]g^{-1}...g[a_k, b_k]g^{-1} \in H$. Now $[ga, b][b, g] = gaba^{-1}g^{-1}b^{-1}bgb^{-1}g^{-1} = gaba^{-1}b^{-1}g^{-1} = g[a, b]g^{-1}$.

10. Take $a, b \in C_G(x)$, then $abx = axb = xab$. Thus $ab \in C_G(x)$. $a^{-1}xa = x = xa^{-1}a$, cancel the $a$ on the left, to get $a^{-1}x = xa^{-1}$ so $a^{-1} \in C_G(x)$. 