1. Find the line that minimizes the least squares error for the points (1, 2), (3, 4), (4, 6).

For this problem are minimizing the error of \( C + Dx \) as compared to the data points. The general method is to place the coefficients of \( C \) and \( D \) into a matrix, one row for each \( x \) value.

\[
A = \begin{bmatrix}
  1 & 1 \\
  1 & 3 \\
  1 & 4
\end{bmatrix}.
\]

Then \( (A^tA) = \begin{bmatrix} 3 & 8 \\ 8 & 26 \end{bmatrix} \). The inverse is \( \frac{1}{14} \begin{bmatrix} 26 & -8 \\ -8 & 3 \end{bmatrix} \). And \( (A^tA)^{-1}A^t = \frac{1}{14} \begin{bmatrix} 18 & 2 & -6 \\ -5 & 1 & 4 \end{bmatrix} \).

Multiplying this by the vector \( \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \) gives \( C = \frac{4}{7} \) and \( D = \frac{9}{7} \) and the best linear approximation is \( \frac{4}{7} + \frac{9}{7}x \).

2. Find the parabola that minimizes the least squares error for the points (0,0), (1,8), (3,8) and (4,20).

Here we use the equation \( C + Dx + Ex^2 \) to model the data, giving matrix

\[
A = \begin{bmatrix}
  1 & 0 & 0 \\
  1 & 1 & 1 \\
  1 & 3 & 9 \\
  1 & 4 & 16
\end{bmatrix}.
\]

Then

\[
(A^tA) = \begin{bmatrix}
  4 & 8 & 26 \\
  8 & 26 & 92 \\
  26 & 92 & 338
\end{bmatrix}.
\]

The inverse is

\[
\frac{1}{90} \begin{bmatrix}
  81 & -78 & 153 \\
  -78 & 169 & -40 \\
  15 & -40 & 10
\end{bmatrix}.
\]

And

\[
(A^tA)^{-1}A^t = \frac{1}{30} \begin{bmatrix}
  27 & 6 & -6 & 3 \\
  -26 & 17 & 23 & -14 \\
  5 & -5 & -5 & 5
\end{bmatrix}.
\]

Multiplying this by the vector \( \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \) gives \( C = 2 \), \( D = \frac{4}{3} \) and \( E = \frac{2}{3} \) and the best quadratic or parabolic approximation is \( 2 + \frac{4}{3}x + \frac{2}{3}x^2 \).

3. Compute the line that minimizes the least squares error for the data in problem 2. Does the line or the parabola more closely match the data?

Here we use the equation \( C + Dx \) to model the data, giving matrix

\[
A = \begin{bmatrix}
  1 & 0 \\
  1 & 1 \\
  1 & 3 \\
  1 & 4
\end{bmatrix}.
\]

Then

\[
(A^tA) = \begin{bmatrix}
  4 & 8 \\
  8 & 26
\end{bmatrix}.
\]

The inverse is

\[
\frac{1}{20} \begin{bmatrix}
  13 & -4 \\
  -4 & 2
\end{bmatrix}.
\]
And

\[(A^tA)^{-1}A^t = \frac{1}{20} \begin{bmatrix} 13 & 9 & 1 & -3 \\ -4 & -2 & 2 & 4 \end{bmatrix}.\]

Multiplying this by the vector \((0, 8, 8, 20)\) gives \(C = 1, D = 4\) and the best linear approximation is \(1 + 4x\).

Note that the parabolic computation includes the case of a line (i.e. when \(E = 0\)), since the parabolic case is not a line, the parabolic approximation must be better.

4. Compute the determinant of

\[A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.\]

Using row reduction:

\[\det A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.\]

5. What are the determinants of \(A^{-1}\) and \(A^3\) where \(A\) is the above matrix?

Since \(\det(A^{-1}) = \frac{1}{\det(A)}\), we have \(\det(A^{-1}) = 1\). Moreover \(\det(A^3) = \det(AAA) = \det(A)\det(A)\det(A) = (\det(A))^3 = 1\). In general we have \(\det(A^n) = (\det(A))^n\) for any integer \(n\).

6. Compute the determinant of

\[\begin{bmatrix} 1 & 3 & -1 \\ 2 & -2 & 4 \\ 1 & 0 & 3 \end{bmatrix}\]

in two ways, by expanding along the first column and along the first row. Check that you get the same result for either expansion.

Expanding along the first column,

\[\begin{vmatrix} 1 & 3 & -1 \\ 2 & -2 & 4 \\ 1 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} -2 & 4 \\ 0 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ -2 & 4 \end{vmatrix} = 1(-6) - 2(9) + 1(10) = -14.\]

Expanding along the first row,

\[\begin{vmatrix} 1 & 3 & -1 \\ 2 & -2 & 4 \\ 1 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} -2 & 4 \\ 0 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 1(-6) - 3(2) - 1(2) = -14.\]