1. Find a basis for the orthogonal complement of \( U = \text{span}\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \} \). Hint: create a matrix \( A \) with row \( A = U \), then \( \text{nul} \ A = U^\perp \).

Let 

\[
A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix}.
\]

Its row reduced echelon form is:

\[
\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}.
\]

Thus a basis for the null space is

\[
\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}.
\]

2. Let \( A \) be an \( m \times n \) matrix whose columns are mutually orthogonal (i.e. each column is orthogonal to every other column). What is \( A^t A \)?

Let \( a_i \) be the vector that is the \( i \)-th column of \( A \). Then the \( i, j \) entry of \( A^t A \) is the product of the \( i \)-th row of \( A^t \) and \( j \)-th column of \( A \). Thus it is \( a_i \cdot a_j \). Since the columns are orthogonal, the result is a diagonal matrix with the entries \( ||a_i||^2 \) down the diagonal.

3. Compute the orthogonal projection of the vector \( \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \) onto the subspace \( U \) with basis is \( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \).

Let 

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

\[
A^t A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.
\]

Thus

\[
(A^t A)^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Then

\[
A(A^t A)^{-1} A^t = \frac{1}{3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}.
\]
Multiplying by \[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}
\]
gives the projection \[
\frac{1}{3}
\begin{bmatrix}
8 \\
3 \\
5 \\
11
\end{bmatrix}.
\]

4. Given the basis \[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
\]
of the subspace \(U\), using the Gram-Schmidt process, find an orthonormal basis for the space.

We take \(b_1 = \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}\). Then \(b_2\) should be parallel to
\[
\begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix} - \begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix} \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix} - \frac{6}{30} \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix} = \frac{1}{5} \begin{bmatrix}
-1 \\
3 \\
-3 \\
1
\end{bmatrix}.
\]

We will take \(b_2 = \begin{bmatrix}
-1 \\
3 \\
-3 \\
1
\end{bmatrix}\). \(b_3\) should be parallel to
\[
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} - \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix} = \begin{bmatrix}
-1 \\
3 \\
-3 \\
1
\end{bmatrix} - \frac{3}{3} \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
2 \\
1 \\
0 \\
-1
\end{bmatrix}.
\]

So we take \(b_3 = \begin{bmatrix}
2 \\
0 \\
-1
\end{bmatrix}\). Then \(c_1 = \frac{1}{\sqrt{30}} \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, c_2 = \frac{1}{\sqrt{20}} \begin{bmatrix}
1 \\
3 \\
-3
\end{bmatrix}\) and \(c_3 = \frac{1}{\sqrt{6}} \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix}\).

5. Using the above basis, what is the orthogonal projection of \[
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]
onto \(U\)? Don’t use the \(A(A^tA)^{-1}A^t\) method.

We need to use the projection formula that is based on the dot products, specifically
\[
\text{proj}_U \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} =
\]
\[
\begin{pmatrix}
1 & 1 \\
0 & \frac{1}{\sqrt{30}} & 2 \\
0 & 3 & \frac{2}{4}
\end{pmatrix}
\begin{pmatrix}
1 & 3 \\
0 & \frac{1}{\sqrt{20}} \\
0 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
0 & \frac{1}{\sqrt{20}} \\
0 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & \frac{1}{\sqrt{6}} \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & \frac{1}{\sqrt{6}} \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & \frac{1}{\sqrt{30}} \\
0 & 3 & \frac{2}{4}
\end{pmatrix}
\begin{pmatrix}
1 & 3 \\
0 & \frac{1}{\sqrt{20}} \\
0 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
0 & \frac{1}{\sqrt{20}} \\
0 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & \frac{1}{\sqrt{6}} \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & \frac{1}{\sqrt{30}} \\
0 & 3 & \frac{2}{4}
\end{pmatrix}
\begin{pmatrix}
1 & 3 \\
0 & \frac{1}{\sqrt{20}} \\
0 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
0 & \frac{1}{\sqrt{20}} \\
0 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & \frac{1}{\sqrt{6}} \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & \frac{1}{\sqrt{30}} \\
0 & 3 & \frac{2}{4}
\end{pmatrix}
\begin{pmatrix}
1 & 3 \\
0 & \frac{1}{\sqrt{20}} \\
0 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
0 & \frac{1}{\sqrt{20}} \\
0 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & \frac{1}{\sqrt{6}} \\
0 & 0 & -1
\end{pmatrix}
\]