1. Find a basis for the row space, column space and null space of
\[
\begin{bmatrix}
3 & 0 & 1 \\
-1 & 1 & 2 \\
0 & -6 & -10
\end{bmatrix}.
\]

The reduced row echelon form of the matrix is
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The thus a basis for the row space is
\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}.
\]

A basis for the column space is
\[
\begin{bmatrix}
3 \\
-1 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
-6
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
-10
\end{bmatrix}.
\]

The basis for the null space is empty since the null space contains just the 0 vector.

2. Use the reduced row echelon form algorithm, solve
\[
\begin{bmatrix}
1 & 0 & 2 & -1 \\
-2 & 2 & 0 & 1 \\
0 & 4 & 8 & -2
\end{bmatrix} \mathbf{x} = \begin{bmatrix}
2 \\
1 \\
10
\end{bmatrix}.
\]

Bringing the augmented matrix to reduced row echelon form:
\[
\begin{bmatrix}
1 & 0 & 2 & -1 & 2 \\
-2 & 2 & 0 & 1 & 1 \\
0 & 4 & 8 & -2 & 10
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 2 & -1 & 2 \\
0 & 2 & 4 & -1 & 5 \\
0 & 4 & 8 & -2 & 10
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 2 & -1 & 2 \\
0 & 2 & 4 & -1 & 5 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 2 & -1 & 2 \\
0 & 1 & 2 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This system has two pivots in the \(x\) and \(y\) columns and two free variables in the \(z\) and \(w\) columns, thus the general solution is
\[
\begin{bmatrix}
2 \\
2 \\
0
\end{bmatrix} + z \begin{bmatrix}
-2 \\
-2 \\
0
\end{bmatrix} + w \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]
3. Find a basis for
\[
\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix} \right\}.
\]
Putting these vectors into the rows of a matrix
\[
A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 0 & -3 & -3 \end{bmatrix}.
\]
Reducing:
\[
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{bmatrix}
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}
\]
Thus \[ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \] is a basis.

4. Consider the set of all 2 by 2 matrices \( M \) such that \( M \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -M^t \). Is this a vector subspace of \( M_{2,2} \)? If so, what is a basis? Hint: think of \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and derive a system of linear equations in the coefficients.

Following the hint, we derive a system of equations in the entries \( a, b, c \) and \( d \) that are satisfied if and only if the matrix is a solution to the given problem. The equation becomes
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t
\]
\[
\begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}.
\]
For this equation is satisfied if and only if the entries on the right are equal to the entries on the left. That is, we have the equations \( b = -a, a = -c, d = -b \) and \( c = -d \). Note that \( b = -a \) and \( c = -a \), so \( d = a \) is consistent with both equation involving \( d \). Thus a matrix is a solution if and only if it is of the form \( \begin{bmatrix} a & -a \\ -a & a \end{bmatrix} \) or \( \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \). But then the solutions are just \( \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \) and since every span is a subspace, the solutions of the equation are a subspace. Also, since the given matrix is not zero, it is also a basis, since it is thus linearly independent.