NAME: ________________________

• Time: you have 50 minutes (12:20-1:10PM).
• Show all your work and justifications.
• The use of calculators or notes is not permitted during the exam.
• You may use both sides of the paper.

________________________________

Academic integrity is expected of all students of Cornell University at all times. In particular, you are not allowed to discuss the exam with the students of the upcoming class (1:25-2:15 session) after you leave this room.
Understanding this, I declare I shall not give, use, or receive unauthorized aid.

SIGNATURE: ________________________

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Problem 1: ___ / 25
Problem 2: ___ / 25
Problem 3: ___ / 25
Problem 4: ___ / 25

Total: ___ / 100
**Problem 1**: Consider the matrix

\[ A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 1 \\ 3 & 1 & 4 & 3 \end{pmatrix}. \]

(a) Find a basis for each of the following subspaces:

- Col(A) (i.e. the column space of A).
- Row(A) (i.e. the row space of A).
- Null(A) (i.e the null space of A).

(b) What is the rank of A?

(c) What is the orthogonal complement of Null(A) in \( \mathbb{R}^4 \)?
Problem 2: Consider the matrix

\[ Q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}. \]

(a) Show that the columns of \( Q \) form an orthonormal basis for the subspace \( W = \text{Col}(Q) \) of \( \mathbb{R}^4 \).

(b) Compute the projection of

\[ b = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \]

onto \( W \).
**Problem 3:** Let $A$ be an $m \times n$ matrix. Recall if the columns of $A$ are independent, the projection matrix onto $W = \text{Col}(A)$ can be computed by

$$ P = A(A^T A)^{-1} A^T. $$

Note that $P$ is a square ($m \times m$) matrix.

(a) If the columns of $A$ are not independent, how would you compute $P$?

(b) Show that for any projection matrix $P$, we have $P^2 = P$.

(c) Let $\det(P) = x$. The following argument suggests that $x$ is always 1, which is incorrect.

$$ P = A(A^T A)^{-1} A^T \quad \text{so} \quad x = \det(A) \frac{1}{\det(A^T) \det(A)} \det(A^T) = 1. $$

What is wrong with this argument?

(d) Show that $x$ is always either 0 or 1.

(\textit{Hint for part (d): use the equality in part (b), look at the determinant of each side, and use the fact that determinant “respects” multiplication.})
Problem 4: What is the volume of the 3-dimensional paralleotope in $\mathbb{R}^4$ generated by

\[
\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.
\]