Problems

Problem 1. The stochastic matrix is \( P = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \).

The situation of the TA being fine now is represented by the probability vector \( v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Then the probability of him being fine after one more solution is \( Pv = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} \).

If there is a 30% chance the TA is sad now, it is represented by the vector \( u = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \). Since \( Pu = \begin{bmatrix} 0.69 \\ 0.31 \end{bmatrix} \), the probability of the TA being fine after one more solution is 69%.

The equilibrium vector \( q \) satisfies the equation \( Pq = q \). Solving the system of equations \((P - I)q = 0\), i.e. finding the null space of \( P - I = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix} \) we see that \( q = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \). Thus eventually TA will be spending 1/3 of his time frustrated. Not too bad.

Problem 2. Suppose we have a linear combination 
\[ c_1(v_1 + 3v_2 - v_3) + c_2(-v_1 - 2v_2 + v_3) + c_3(v_2 + 2v_3) = 0. \]

We would like to know whether this combination can be non-trivial, i.e. whether we can find not all zero scalars \( c_1, c_2, c_3 \) making the equation above true. We can re-write this equation as
\[ (c_1 - c_2)v_1 + (3c_1 - 2c_2 + c_3)v_2 + (-c_1 + c_2 + 2c_3)v_3 = 0 \]

But we know that the vectors \( v_1, v_2, v_3 \) are linearly independent. Therefore for the equation above to be true we must have \( c_1 - c_2 = 0, 3c_1 - 2c_2 + c_3 = 0 \) and \( -c_1 + c_2 + 2c_3 = 0 \). For which \( c_1, c_2, c_3 \) can this be?

In other words, we are really asking for solutions of the homogeneous system of equations
\[
\begin{bmatrix}
1 & -1 & 0 \\
3 & -2 & 1 \\
-1 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix}
\]

Row reduce, see that each column is pivotal, so there are no non-trivial solutions, so \( c_1 = c_2 = c_3 = 0 \), and so the required vectors are linearly independent.

Problem 3. We have \( A - 2I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \). This matrix has non-trivial null space (for example, because \( \det(A - 2I) = 0 \)). So \( \lambda = 2 \) is an eigenvalue.

Problem 4. \( \lambda = 1 \) and \( \lambda = 3 \) are the only eigenvalues of \( A \).

Let’s find an eigenvector for \( \lambda = 1 \). We need to find \( v \) s.t. \( Av = \lambda v \), i.e. we are looking for null space of \( A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \). Row reduce, find that the null space consists of vectors of the form \( x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \). So for example, \( \begin{bmatrix} 2 \\ -2 \end{bmatrix} \) is an eigenvector. Any other non-zero multiple of this one would work.

For \( \lambda = 3 \) we can take \( v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

Problem 5. For a number \( \lambda \) to be an eigenvalue, we need the matrix
\[
A - \lambda I = \begin{bmatrix}
1 - \lambda & 3 & -3 \\
0 & 2 - \lambda & 1 \\
0 & 0 & 3 - \lambda
\end{bmatrix}
\]
to have non-trivial null-space. In other words, we want to have some free variables in the corresponding homogeneous system, or equivalently we want to have some non-pivotal columns. If $\lambda$ is not equal 1, 2, 3 then every column would be pivotal. If $\lambda = 1, 2$ or 3, one of the columns would be non-pivotal. So the eigenvalues are 1, 2, 3.

Since eigenvalues are all distinct, the corresponding eigenvectors are linearly independent.

**Problem 6.** Number 0 is an eigenvalue means that for some non-zero vector $v \neq 0$ one has $Av = 0 \cdot v = 0$. Thus $A$ has non-trivial null space, and so can’t be invertible.

**Problem 7.** The matrix $A$ is invertible, hence non of the eigenvalues $\lambda_1, \ldots, \lambda_n$ are zero. Now, if $Av = \lambda v$ and $A$ is invertible, we can multiply $Av = \lambda v$ by $A^{-1}$ and get $v = \lambda A^{-1}v$, and so $A^{-1}v = \frac{1}{\lambda} v$. So $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ are the eigenvalues of $A^{-1}$.

**Problem 8.** Equilibrium vector is an eigenvector corresponding to eigenvalue 1.

**Problem 9.**
- False
- False
- True
- False

**Problem 10.** Transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ reflecting with respect to the line $x = y$ preserves any vector on this line. So $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector, and since it is being preserved, i.e. $T(v) = v$, the eigenvalue is 1. The other line that is preserved is the line $x = -y$, since it is perpendicular to $x = y$. Each vector $u$ on the line $x = -y$ is reflected, i.e. gets mapped to $-u$. So for example $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector, with the eigenvalue $\lambda = -1$.

Transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ rotating around the $y$-axis by the angle $\pi/2$ preserves the $y$-axis, so non-zero vector on this axis (for example $v = [0, 1, 0]^T$) will be eigenvectors with the eigenvalue 1.

**Problem 11.** Matrix $A$ is called nilpotent if $A^n = 0$ for some number $n$. Give an example of a $3 \times 3$ non-zero nilpotent matrix. What are the eigenvalues of any nilpotent matrix $A$?

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is non-zero nilpotent, since $A^2 = 0$. If $Av = \lambda v$ for some non-zero $v \neq 0$, then $A^2v = A(\lambda v) = \lambda^2 v$, $A^3v = \lambda^3 v$, et.c. Since $A$ is nilpotent, at some step we get $A^n = 0$, and so we will be looking at the equation $0 = \lambda^n v$. Since $v \neq 0$, then $\lambda^n = 0$, and so $\lambda = 0$. Thus the only eigenvalue nilpotent matrix can have is 0.