Recitation 10

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Problem 1. To find the eigenvectors and eigenvalues we do the same thing we usually do. The characteristic equation is \( \lambda^2 - 8\lambda + 17 = 0 \), which gives \((\lambda - 4)^2 = -1\), and hence \( \lambda_{1,2} = 4 \pm i \) are the two eigenvalues. Notice that they are complex conjugates of each other.

Take \( \lambda = 4 + i \). Then \( A - \lambda I = \begin{bmatrix} 1 - i & -2 \\ 1 & -1 - i \end{bmatrix} \sim \begin{bmatrix} 1 & 1 - i \\ 0 & 0 \end{bmatrix} \). Thus, when solving the system \((A - (4 + i)I)v = 0\) the variable \( x_2 \) is a free variable, and \( x_1 = (1 + i)x_2 \). Therefore eigenvectors for \( \lambda = 4 + i \) are of the form \( v_1 = x_2 \begin{bmatrix} 1 + i \\ 1 \end{bmatrix} \). We can take \( v_1 \) to be \( v_1 = \begin{bmatrix} 1 + i \\ 1 \end{bmatrix} \).

We don’t really have to compute eigenvectors for the other eigenvalue \( \lambda = 4 - i \), since we know (at least I hope we know by that time) that for a real matrix \( A \), eigenvectors corresponding to conjugate eigenvalues are complex conjugate (it’s a really easy fact. Exercise: make sure you know how to prove it). So, the second eigenvector \( v_2 \), for \( \lambda = 4 - i \), can be taken to be \( v_2 = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix} \).

Note: if we would be after diagonalizing the matrix \( A \), we would have

\[
A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} = [v_1 v_2]D[v_1 v_2]^{-1} = \begin{bmatrix} 1 + i & 1 - i \end{bmatrix} \begin{bmatrix} 4 + i & 0 \\ 0 & 4 - i \end{bmatrix} \begin{bmatrix} 1 + i & 1 - i \\ 1 & 1 \end{bmatrix}^{-1}
\]

However, this is not what the question is really asking. To find the required matrices \( P \) and \( C \), we need some knowledge from the book (I mean D.C.Lay’s book, not the Book). We need to pick one of the eigenvalues and stick to it. Let’s take \( \lambda = 4 + i \). Then we read \( C \) from this eigenvalue: if the eigenvalue is of the form \( a + bi \), then \( C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \). So in our case we have

\[
C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}
\]

To find \( P \), we read it off the eigenvector \( v_1 \) for our fixed eigenvalue as \( P = [\text{Re}(v_1), \text{Im}(v_1)] \). So we have

\[
P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}
\]

Thus we have

\[
A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} = PCP^{-1} = P \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}
\]

We could fix the other eigenvalue \( \lambda_2 = 4 - i \) and do the same thing using this value and the eigenvector \( v_2 \) for this eigenvalue. In this case we would get a different (but equivalent) decomposition

\[
A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} = P_2C_2P_2^{-1} = P \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1}
\]

Problem 2. We do the same thing you did in the first prelim. Suppose some linear combination of vectors \( v_1, v_2 - v_3, v_1 + v_2 + v_3 \) is zero, i.e.

\[
x_1 v_1 + x_2 (v_2 - v_3) + x_3 (v_1 + v_2 + v_3) = 0
\]

We want to prove that necessarily \( x_1 = x_2 = x_3 = 0 \). The equation above reads

\[
(x_1 + x_3)v_1 + (x_2 + x_3)v_2 + (-x_2 + x_3)v_3 = 0
\]
Since \(v_1, v_2, v_3\) are independent by the assumption, we must have \(x_1 + x_3 = 0, x_2 + x_3 = 0\) and \(-x_2 + x_3 = 0\). In other words, we are solving the system of equations

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Row reduce, see that there are three pivots, so the system only has trivial solution, so \(x_1 = x_2 = x_3 = 0\), and that’s what we were shooting for. So \(C = \{v_1, v_2 - v_3, v_1 + v_2 + v_3\}\) is a basis of \(H\).

To find the matrix of \(T\) relative to \(B = \{v_1, v_2, v_3\}\), we need to see where basis vectors \(v_1, v_2, v_3\) go, find their coordinates (weights) in the basis \(B\), and that would be columns of our matrix. We have

\[
T(v_1) = v_3 - v_2, \quad \text{and the coordinates of } T(v_1) \text{ in } B \text{ are } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}. \quad \text{Since } T(v_2) = v_1 - 2v_2 + v_3, \text{ the second}
\]

column of the matrix we are looking for is \(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\). Similarly the third column is \(\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}\). So the matrix is

\[
A = [T]_B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & -2 & -1 \\ 1 & 1 & -2 \end{bmatrix}
\]

To find the matrix of \(T\) in the new basis \(C\) knowing its matrix \(A\) in the old basis, we need to find where basis vectors from the new basis \(C\) go, and find their weights in terms of the new basis. These weights will be the columns of our new matrix. Effectively it can be done by finding the matrix \(P\) expressing the new basis in terms of the old one, and using the formula \(M = P^{-1}AP\).

In our case, since \(C = \{v_1, v_2 - v_3, v_1 + v_2 + v_3\}\), \(P\) is just

\[
P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}
\]

Then we have

\[
M = [T]_C = P^{-1}AP = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 3 \\ -1 & -2 & -1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}
\]

Please, finish the calculation yourself.

To find the rank of \(T\) we can use any matrix (relative to any basis) of \(T\) and find its rank. Using \(A\), for example, we row reduce and see

\[
A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & -2 & -1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -20 & 1 & 3 \\ 0 & -1 & -3 & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -20 & 1 & 3 \\ 0 & 0 & 0 & \end{bmatrix}
\]

There are two pivots, so the rank is also 2.

**Problem 3.** I hope by now proving that the two sets form a basis of \(\mathbb{R}^2\) is not a problem.

We know that relative to the basis \(B = \{v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}\}\) the matrix of \(T\) is \(A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\). We want to find the matrix of \(T\) relative to the basis \(C = \{w_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ -8 \end{bmatrix}\}\). Let’s fix notation

\[
P = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} \quad \text{and } \quad Q = \begin{bmatrix} -2 & 3 \\ 5 & -8 \end{bmatrix}.
\]

We can use the same old strategy. We need to find where vectors from \(C\) go under \(T\), and find the weights of the resulting vectors in terms of \(C\). More efficiently (but really the same), we need to represent the new basis in terms of the old one, it will be given by some matrix, say \(R\), and then the matrix in the new basis is just \(M = R^{-1}AR\).

Let’s represent the new basis \(w_1, w_2\) via the old one \(v_1, v_2\). We are looking for scalars \(a, b, c, d\) s.t.

\[
w_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix} = av_1 + bv_2 = a\begin{bmatrix} -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \text{and } \quad w_2 = \begin{bmatrix} 3 \\ -8 \end{bmatrix} = cv_1 + dv_2 = c\begin{bmatrix} -1 \\ 2 \end{bmatrix} + d\begin{bmatrix} -1 \\ 3 \end{bmatrix}.
\]

In other words, we need to find \(a, b, c, d\) such that

\[
\begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 5 & -8 \end{bmatrix}
\]
Thus the matrix \[
\begin{bmatrix}
a & c \\
1 & 1
\end{bmatrix}
\] is just \( P^{-1}Q = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \). Let’s call it \( R \). This matrix represents the new basis in terms of the old basis. But that’s good news! We can now use the general formula: if \( R \) is the new basis in terms of the old one, and \( A \) is the matrix of \( T \) in the old basis, then the matrix \( M \) of \( T \) in the new basis is just \( M = R^{-1}AR \). Thus we have

\[
M = [T]_c = R^{-1}AR = \begin{bmatrix}
1 & -2 \\
1 & -1
\end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ -4 & 7 \end{bmatrix}
\]

Kind of ugly...

**Problem 4.** We have \( W = \{ w \in \mathbb{R}^3 \mid w \cdot v = 0 \} \) consists of vectors \( w = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) such that \( w \cdot v = 0 \), i.e. \( a - 2b + c = 0 \). This is an equation of a plane, so it is a subspace.

**Note:** \( W \) is the null space of \( A = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \). So to find basis of \( W \) we are really looking for a basis of the null space of \( A = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \). I hope you know how to do that.

**Problem 5.** If \( x \in W \) and \( x \in W^\perp \), it means that \( x \cdot x = 0 \). But this can only happen when \( x = 0 \).

**Problem 6.** You just use the projection formula. If \( y = \tilde{y} + r \) with \( \tilde{y} \in Span(u) \) and \( r \in Span(u)^\perp \), then
\[
\tilde{y} = \frac{u \cdot y}{u \cdot u} u = \frac{8 - 21}{10 + 19} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 \\ -7 \end{bmatrix}.
\]

Then \( r = y - \tilde{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix} \). Also kind of ugly.

**Problem 7.** Since \( u \cdot v = 1 \cdot 2 + (-2) \cdot 1 = 0 \), the vectors are orthogonal. However, since \( u \cdot u = 5 \neq 1 \) and \( v \cdot v = 5 \neq 1 \), they are not orthonormal. So we normalize to get \( \{ \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \} \).

**Problem 8.** We have \( U^T U = I \) so \( U \) is invertible.

**Problem 9.** We have \( UV(UV)^T = UVV^TU^T = UIU^T = UU^T = I \) so \( UV \) is also orthogonal.

**Problem 10.** Let \( u \in \mathbb{R}^n \) be a non-zero vector, and denote \( L = Span(u) \). Prove that the map \( T : \mathbb{R}^n \to L \) given by \( T(y) = proj_L(y) \) is a linear transformation. Just use the formula \( T(y) = \frac{u^T u}{u^T u} u \), and using the properties of dot product check that \( T \) satisfies the definition of a linear transformation.