IDEALS DEFINING UNIONS OF MATRIX SCHUBERT VARIETIES

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Abstract. This note computes a Gröbner basis for the ideal defining a union of matrix Schubert varieties. Moreover, the theorem presented will work for any union of schemes defined by northwest rank conditions. This provides a means of intersecting a large class of determinantal ideals.

Introduction

We compute a Gröbner basis for the ideal defining a union of schemes defined by northwest rank conditions with respect to the “antidiagonal term order.” By this we mean any scheme whose defining equations are of the form “all $k \times k$ minors in the northwest $i \times j$ submatrix of a matrix of variables, where $i$, $j$, and $k$ can be filled in with varying values. On the algebraic side, this paper provides access to a larger set of examples of determinental varieties. On the geometric side, we have computed a generating set for the ideal defining a union of matrix Schubert varieties.

Let $B_-$ (respectively $B_+$) denote the group of invertible lower triangular (respectively upper triangular) matrices. Let $M = (m_{i,j})$ be a matrix of variables. In what follows $\pi$ will be a (possibly partial) permutation, written in one-line notation, $\pi(1) \ldots \pi(n)$.

Definition. A matrix Schubert variety $X_\pi$ is the closure $B_-\pi B_+$ in the space of all matrices where $\pi$ is a (possibly partial) permutation matrix.

Matrix Schubert varieties associated to honest permutations are the closures of the lifts of the corresponding Schubert varieties in $B_- \setminus GL_n$ and were introduced in [Ful92]. Projecting \{full rank matrices\} $\rightarrow B_- \setminus GL_n$ sends matrix Schubert varieties corresponding to (honest) permutations to Schubert varieties.

Definition. In this paper, we will denote the greatest term with respect to some term order of a polynomial $f$ by $\text{init } f$ and the initial ideal $\langle \text{init } f : f \in I \rangle$ will be denoted by $\text{init } I$. A generating set $f_1, \ldots, f_n$ for an ideal $I$ is a Gröbner basis for $I$ if $\text{init } I = \langle \text{init } f_1, \ldots, \text{init } f_n \rangle$.

Definition. The rank matrix of a (partial) permutation $\pi$, denoted $r(\pi)$, gives in each cell $r_{i,j}$ the rank of the $i \times j$ northwest-justified submatrix of the permutation matrix for $\pi$.

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For example, the rank matrix of 15432 is
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 5
\end{pmatrix}.
\]

**Theorem 1** (Fulton). Matrix Schubert varieties have radical ideal $I(X_{\pi}) = I_{\pi}$ given by determinants representing conditions given in the rank matrix $r(\pi)$, that is, the $(r(\pi)_{i,j} + 1) \times (r(\pi)_{i,j} + 1)$ determinants of the northwest $i \times j$ submatrix of a matrix of variables.

**Definition.** Hereafter we call these determinants or the analogous determinants for any ideal generated by northwest rank conditions the **Fulton generators**.

**Definition.** The antidiagonal of a matrix is the diagonal series of cells in the matrix running from the most northeast to the most southwest cell. The antidiagonal term (or antidiagonal) of a determinant is the product of the entries in the antidiagonal.

For example, the antidiagonal of \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is the cells occupied by $b$ and $c$, and correspondingly, in the determinant $ad - bc$ the antidiagonal term is $bc$. Typically these will be denoted $A$ or $B$ in what follows, with the notions of sets of cells and monomials interchanged freely.

**Theorem 2** ([KM05]). The Fulton generators for the matrix Schubert variety $I_{\pi}$ form a Gröbner basis for $I_{\pi}$ under any term order that picks the antidiagonal term as the initial term from each determinant. Further, the corresponding initial ideal init $I_{\pi}$ is the Stanley-Reisner idea of a shellable simplicial complex known as the “pipe dream complex.”

**Theorem 3** ([?]). If $\{I_i : i \in S\}$ are ideals corresponding generated by northwest rank conditions then $\text{init} (\cap_{i \in S} I_i) = \cap_{i \in S} (\text{init} I_i)$.

**Preliminaries**

**Lemma 1.** If $J \subseteq K$ are homogeneous ideals in a polynomial ring such that init $J = \text{init} K$ then $J = K$.

**Proof.** See [KM05]. \(\square\)

**Lemma 2.** Let $I_A$ and $I_B$ be ideals that define varieties of northwest rank conditions and let $g_A \in I_A$ and $g_B \in I_B$ be determinants with antidiagonals $A$ and $B$ respectively such that $A \cup B$ is the antidiagonal of a submatrix $M$. Then $\det(M)$ is in $I_A \cap I_B$. 

Proof. Let $V_M = V(\det(M))$, $V_A = V(I_A)$ and $V_B = V(I_B)$. It is enough to show that $V_A \subseteq V_M$ and $V_B \subseteq V_M$ and, by relabeling just that $V_A \subseteq V_B$.

Let $n$ be the number of rows (also the number of columns) in the submatrix $M$. Assume that the antidiagonal $A$ is of length $r + 1$, with left-most dot in column $t + 1$ (hence in row $n - t$) of $M$ and right-most dot in column $c$. Notice $c \geq (t + 1) + (r + 1)$, with equality if $A$ occupies a continuous set of columns, so matrices in $V_A$ have rank at most $r$ in the northwest $(n - t) \times (t + r + 2)$ and hence have rank at most $r + (n - t - r - 2) = n - t - 2$ in the northwest $(n - t) \times n$, as we can add at most one to the rank in each additional column. Further, by the same principle, moving down $t$ rows, the northwest $n \times n$, i.e. the whole matrix, has rank at most $n - t - 2 + t = n - 2$, hence has rank at most $n - 1$ and so is in $V_M$.

![Figure 1](image-url)  

**Figure 1.** The elements of the antidiagonal of $g_A$ in the proof are shown with filled dots, while the entries of the antidiagonal of $M$ that are only in the antidiagonal of $g_B$ are shown with unfilled dots. The rank conditions $r + n - c$ and $r + n - c + t$ are those implied by the rank condition $r$ met by all matrices in $V_A$.

□

**Formula**

The following will be elements of the Gröbner basis for $\cap I_i$ where each $I_i$ is an ideal generated by northwest rank conditions:
**Definition.** Fix a set of antidiagonals \( A_1, \ldots, A_n \) such that \( A_i \) is the antidiagonal of a Fulton generator of \( I_{\pi_i} \). The generator \( g_{A_1, \ldots, A_n} \) is given by

\[
g_{A_1, \ldots, A_n} = \sum (-1)^{\text{sign}(f)} \prod_{b \in \cup A_i} m_{\text{row}(b), f(b)}
\]

where the sum is over all possible functions

\[
f : \{\cup A_i\} \to \text{columns}(\cup A_i)
\]

subject to the restrictions:

1. For each column \( c \), \( |f^{-1}(c)| = |c \cap (\cup A_i)| \). Equivalently, \( g_{A_1, \ldots, A_n} \) is homogeneous under the \( T \)-action on the right.
2. \( f \) is injective on each \( A_i \).
3. For each box \( b \in A_i \), \( f(b) \leq \max\{c : c \text{ is a column of a set } S \text{ containing } A_i\} \), where \( S \) is any collection of boxes such that if row(\( a \)) \( \geq \) row(\( b \)) then column(\( a \)) \( \leq \) column(\( b \)). and \( S \) contains no two boxes in the same row.

Here \( \text{sign}(\bullet) \) is an extension of the notion of the sign of a permutation: order the boxes containing dots first by row and then by column, working west to east, and similarly order the occupied columns from left to right. Use this ordering to make a partial permutation corresponding to the function \( \bullet \). The sign of this partial permutation (the parity of the number of boxes in its diagram) will be \( \text{sign}(\bullet) \).

**Theorem 4.**

\[
\langle g_{A_1, \ldots, A_n} : A_i \text{is an antidiagonal of a Fulton generator of } I_{\pi_i}\rangle = \cap_{i=1}^n I_{\pi_i}
\]

**Examples**

![Diagram](image)

**Figure 2.** The generator for these three antidiagonals would be

\[
\begin{array}{ccc}
m_{1,1} & m_{1,2} & m_{1,3} \\
m_{2,1} & m_{2,2} & m_{2,3} \\
m_{3,2} & m_{3,3} & m_{3,4}
\end{array}
\]

\[
\begin{array}{ccc}
m_{4,2} & m_{4,3} \\
m_{5,2} & m_{2,2}
\end{array}
\]
Figure 3. The generator for these three antidiagonals would be

\[
\begin{vmatrix}
    m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\
    m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\
    m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\
    m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4}
\end{vmatrix}
\]

Figure 4. The generator for these three antidiagonals would be

\[
\begin{vmatrix}
    m_{1,1} & m_{1,2} \\
    m_{2,1} & m_{2,2}
\end{vmatrix}
\]

\[
\begin{vmatrix}
    m_{2,1} & m_{2,2} & m_{2,4} \\
    m_{3,1} & m_{3,2} & m_{3,4} \\
    m_{4,1} & m_{4,2} & m_{4,3}
\end{vmatrix}
\]

\[
+\begin{vmatrix}
    m_{1,1} & m_{1,2} & m_{1,4} \\
    m_{2,1} & m_{2,2} & m_{2,3} \\
    m_{4,1} & m_{4,2} & m_{4,4}
\end{vmatrix}
\]

\[
-\begin{vmatrix}
    m_{1,1} & m_{1,5} \\
    m_{5,1} & m_{5,2} \\
    m_{5,5}
\end{vmatrix}
\]

\[
-\begin{vmatrix}
    m_{1,1} & m_{1,2} & m_{1,4} \\
    m_{2,1} & m_{2,2} & m_{2,3} \\
    m_{3,1} & m_{3,2} & m_{3,4} \\
    m_{4,1} & m_{4,2} & m_{4,3} \\
    m_{4,4}
\end{vmatrix}
\]

\[
-\begin{vmatrix}
    m_{1,1} & m_{1,5} \\
    m_{5,1} & m_{5,5}
\end{vmatrix}
\]

Proof of Theorem 4

Theorem 4 follows from the below list of lemmas and the above more general lemmas. Most importantly, it follows from the key:

**Lemma 3.** \( g_{A_1, \ldots, A_n} \in I_i \) for each \( i \).

**Proof.** Fix \( i \). We will show that \( g_{A_1, \ldots, A_n} \in I_i \). Let \( S \) be the antidiagonal containing \( A_i \) which contains the column \( c \) which achieves the maximum from condition 3. That is, \( S \) is a collection of boxes containing \( A_i \) such that if \( \text{row}(a) \geq \text{row}(b) \) then \( \text{column}(a) \leq \text{column}(b) \) and \( S \) contains at most 1 box per row. Notice that \(|S| \geq |A_i|\). Also, note that the determinant
determinant with antidiagonal term \( S \) is in \( I \) by a previous lemma. For each \( f \) from the algorithm, set \( S' = f(S) \). By condition 3 \( \max\{c' : c' \text{ is a column of } S'\} = c \). Then,

\[
g_{A_1,\ldots,A_n} = \sum (-1)^{\text{sign}(f)} \prod_{b \in \bigcup A_i} m_{\text{row}(b), f(b)}
\]

\[
= \sum_{S'} \left( \sum_{f \text{ s.t. } f(S) = S'} (-1)^{\text{sign}(f)} \prod_{b \in S} m_{\text{row}(b), f(b)} \right) \left( \sum_{f' \text{ s.t. } f'(S) = S'} (-1)^{\text{sign}(f')} \prod_{b \in S^c} m_{\text{row}(b), f(b)} \right)
\]

The left factor in each summand is a determinant of the rows of \( S \) and the columns given by \( S' \) which are, by construction, at most as large as the columns in \( S \) and hence in any ideal generated by northwest rank conditions and containing a determinant with antidiagonal \( S \). \qed

Therefore, \( I \subseteq \cap_{i=1}^n I_{\pi_i} \). Note that there are many such \( g_{A_1,\ldots,A_n} \) generically \( \prod_{i=1}^n \) (number of generators of \( I_{\pi_i} \)) many.

**Lemma 4.** init \( g_{A_1,\ldots,A_n} = \text{the union of the antidiagonals } A_1 \cup \cdots \cup A_n \).

**Proof.** Let \( \iota \) be the function that sends every dot to its original location. We claim that

\[ \text{init } g_{A_1,\ldots,A_n} = (-1)^{\text{sign}(\iota)} \prod \] m_{r,i(r,v)} \] . We show this by induction on the number of places where an arbitrary function \( f \) meeting the requirements above differs from \( \iota \). If \( f \) differs from \( \iota \) in exactly 2 places then after canceling variables that are the same we are left with a 2 \times 2 determinant. \( \iota \) corresponds to the antidiagonal. Assume the result is true for any \( f \) that differs from \( \iota \) in 2m places. Then, by the same argument as above, an \( f \) that differs in an additional 2 places provides an even smaller term, so the original \( \iota \) is the largest term. \qed

**Lemma 5.** The generating set \( \{g_{A_1,\ldots,A_n}\} \) given above is a Gröbner basis for \( I \).

**Proof.** init \( I = \langle \text{init } g_{A_1,\ldots,A_n} \rangle \) where the \( g_{A_1,\ldots,A_n} \) s are the generators given by the above plan follows from lemma 4. \qed

Note that we have not used any facts about the \( I_{\pi_i} \) besides that they are ideals corresponding to (reduced, but not necessarily irreducible) schemes that are determined by northwest rank conditions.

**References**
