Notions of span and linear independence allow now to define basis of a vector space. Let $V$ be a vector space. Its vectors $v_1, \ldots, v_k$ are called a basis of $V$ if they are linearly independent and span $V$.

For example, vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^3$. Indeed, they are linearly independent: if $ae_1 + be_2 + ce_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ equals 0, then $a = b = c = 0$. And they span $\mathbb{R}^3$ because any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be represented as the linear combination $xe_1 + ye_2 + ze_3$. The basis $\{e_1, e_2, e_3\}$ is called the standard basis of $\mathbb{R}^3$. Similarly, vectors 

$$
\begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0 \\
0 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
0 \\
0 \\
1
\end{bmatrix}, \quad \ldots, \quad
\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
0 \\
1
\end{bmatrix}
$$

form the standard basis of $\mathbb{R}^n$.

**Exercises.**

1°. Denote by $\mathbb{R}_n$ the vector space of all $1 \times n$ matrices. What would be the standard basis in it?

2°. Let $\text{Mat}(n, m)$ be the vector space of all $n \times m$ matrices ($\text{Mat}(n, 1) = \mathbb{R}^n$, $\text{Mat}(1, n) = \mathbb{R}_n$). Denote by $E_{ij}$ the matrix which $(i, j)$ entry is 1 and all other entries are 0. Show that all $E_{ij}$ form a basis of $\text{Mat}(n, m)$.

3°. Consider the vector space $\text{Pol}(n)$ of all polynomials of degree $\leq n$. Prove that the polynomials $t^n, t^{n-1}, \ldots, t^2, t, 1$ form a basis in $\text{Pol}_n$. [It is called the standard basis.]

All these examples illustrate that the vectors spaces we usually consider have obvious bases. There are many other, not that obvious, vector sets which form a basis.

**Example.** Find out whether the vectors

$$
\begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
0 \\
3
\end{bmatrix}
$$

form a basis in $\mathbb{R}^3$.

These vectors are linearly independent since the $3 \times 3$ matrix

$$
A = [v_1 \quad v_2 \quad v_3] =
\begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 0 \\
0 & 1 & 3
\end{bmatrix}
$$
has the nonzero determinant (it is 1). Now check if the span of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) equals \( \mathbb{R}^3 \).

For this, take an arbitrary vector \( \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) and solve the linear equation \( x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{v} \), which is equivalent to the system

\[
\begin{align*}
x + y + 2z &= a, \\
2x + y &= b, \\
y + 3z &= c,
\end{align*}
\]

or \( A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{v} \) where matrix \( A \) is as above. Since \( \det(A) \neq 0 \), \( A \) is invertible. Hence the linear system has the solution \( A^{-1}\mathbf{v} \). Thus we have shown that \( \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3 \).

In this example we used only that \( \det(A) \neq 0 \). If \( \det(A) = 0 \), the vectors would be linearly dependent by a theorem from last lecture, and therefore would not be a basis. Essentially, we proved

**Theorem.** Vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) form a basis of \( \mathbb{R}^n \) if and only if the determinant of the \( n \times n \) matrix \( A = [\mathbf{v}_1 \; \mathbf{v}_2 \; \cdots \; \mathbf{v}_n] \) is nonzero.

Note that this theorem concerns only the case of \( n \) vectors in \( \mathbb{R}^n \). There is a natural question: is it possible that \( m \) vectors form a basis of \( \mathbb{R}^n \) and \( m \neq n \)? The answer is NO, all bases consist of the same number of vectors. To illustrate it, consider the space \( V = \mathbb{R}^2 \). Its standard basis consists of two vectors \( \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

Prove that there are no bases in \( \mathbb{R}^2 \) consisting of one or three vectors. The former is impossible because a vector \( \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \) can’t span \( \mathbb{R}^2 \). Indeed, otherwise \( \mathbf{e}_1 = c\mathbf{v} \) for some number \( c \), which implies \( b = 0 \); but then \( \mathbf{e}_2 \) is not of the form \( d\mathbf{v} \), hence not in the span of \( \mathbf{v} \). Now consider the case of three vectors \( \mathbf{u}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \), \( \mathbf{u}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \), \( \mathbf{u}_3 = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \). Of course, they may span \( \mathbb{R}^2 \). But it still can’t be a basis since the linear independence fails (that is, there is a nontrivial linear combination of \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \) which equals 0). This is because the homogeneous equation \( x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3 = 0 \) has infinitely many solutions (its coefficient matrix is a \( 2 \times 3 \) matrix, and there necessarily will be a free variable). This argument can be generalized to any vector space, not only \( \mathbb{R}^2 \). We have:

**Theorem.** If \( \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \) and \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \) are bases of a vector space \( V \), then \( n = m \).

**Example.** Find out whether the matrices

\[
\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \; \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, \; \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}
\]

are linearly independent.
form a basis of Mat(2, 2).

We know (see the exercise above) that there is a basis consisting of 4 vectors $E_{11}, E_{12}, E_{21}, E_{22}$. Then the matrices above do not constitute a basis, because their amount is 3, which is not equal to 4.

The number of elements in a basis of $V$ is called the dimension of $V$ and denoted by $\dim V$. This is a correct definition since by the theorem all bases consist of the same number of vectors. For example, $\dim \mathbb{R}^n = \dim \mathbb{R}^n = n$, $\dim \text{Pol}(n) = n + 1$, $\dim \text{Mat}(m,n) = mn$. A vector space $V$ is called finite-dimensional, if it has a basis consisting of finite number of vectors (and then this number equals $\dim V$). It is possible, however, that $V$ has no such a basis, e. g. the space of all polynomials (it has a basis of infinite number of vectors: $1, t, t^2, t^3, \ldots$). In this case $V$ is called infinite-dimensional, and $\dim V = \infty$. We will usually deal with finite-dimensional vector spaces.

Note that any linearly independent set $S = \{u_1, u_2, \ldots, u_k\}$ of vectors in $V$ can be complemented to a basis. Indeed, if $\text{Span}(S) = V$, then $S$ is already a basis. If $\text{Span}(S)$ is less than $V$, then take a vector $u_{k+1}$ not containing in $\text{Span}(S)$, and add it to the set $S$. Now we have $S = \{u_1, \ldots, u_k, u_{k+1}\}$ and it is linearly independent (why?). We continue adding vectors to $S$ until $\text{Span}(S) = V$. Then we conclude that $S$ is a basis of $V$. To illustrate this, consider $V$ the space of all traceless $2 \times 2$ matrices, i. e. matrices of the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

and $S$ consisting of a matrix $u_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Take a matrix $u_2$ not in the span of $S$, that is not of the form $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. Say, $u_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Now the span of $S = \{u_1, u_2\}$ consists of matrices $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ for all $b, c$. It is not $V$ yet, e. g. it does not contain the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Denote this matrix by $u_3$ and add to $S$. Then $\text{Span}(S)$ consist of all matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$. Hence $\text{Span}(S) = V$ and $S = \{u_1, u_2, u_3\}$ is a basis of $V$. In particular, $\dim V = 3$.

**Theorem.** Let $V$ be a vector space of dimension $n$. Then any linearly independent set $S$ of $n$ vectors is a basis in $V$.

**Proof.** We only need to show that $\text{Span}(S) = V$. By the argument above, $S$ can be complemented to form a basis of $V$. But then it will have more than $n$ vectors, which can’t be a basis. So, $S$ is already a basis. $\square$

**Example.** Do polynomials

$$u = 2t^2 - t + 1, \ v = -t^2 + 3t + 2, \ w = 5t^2 - 1$$

form a basis of $\text{Pol}(2)$?
Since \( \dim \text{Pol}(2) = 3 \) and we have three polynomials, the last theorem says, \( u, v, w \) form a basis if and only if they are linearly independent. Let’s check it. For this we need to solve the equation \( xu + yv + zw = 0 \), or \( (2x - y + 5z)t^2 + (-x + 3y)t + (x + 2y - z) = 0 \), which leads to the system

\[
\begin{align*}
2x - y + 5z &= 0, \\
-x + 3y &= 0, \\
x + 2y - z &= 0,
\end{align*}
\]

The determinant of the coefficient matrix is \(-30\), which is nonzero. Hence the system has only the trivial solution \( x = y = z = 0 \). Therefore, \( u, v, w \) are linearly independent and form a basis.