Lec 17: Inverse of a matrix and Cramer’s rule

We are aware of algorithms that allow to solve linear systems and invert a matrix. It turns out that determinants make possible to find those by explicit formulas. For instance, if $A$ is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}. \quad (1)$$

Note that the $(i, j)$ entry of matrix (1) is the cofactor $A_{ji}$ (not $A_{ij}$!). In fact the entry is $A_{ji} \det(A)$ as we multiply the matrix by $\frac{1}{\det(A)}$. [We can divide by $\det(A)$ since it is not 0 for an invertible matrix.] Curiously, in spite of the simple form, formula (1) is hardly applicable for finding $A^{-1}$ when $n$ is large. This is because computing $\det(A)$ and the cofactors requires too much time for such $n$. Notice that $\det(A)$ can be found as soon as we know the cofactors, because of the cofactor expansion formula.

Example. Find the inverse, if it exists, for

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & -1 \\ 4 & 0 & 1 \end{bmatrix}.$$ 

We have:

$$A_{11} = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3, \quad A_{12} = -\begin{vmatrix} -2 & -1 \\ 4 & 1 \end{vmatrix} = -2, \quad A_{13} = \begin{vmatrix} -2 & 3 \\ 4 & 0 \end{vmatrix} = -12.$$

Find the determinant by the expansion along the first row:

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 0 \cdot 3 + 1 \cdot (-2) + 2 \cdot (-12) = -26.$$ 

Since $\det(A) \neq 0$, we conclude that $A$ is invertible, and we can continue computing cofactors:\footnote{If the determinant were 0, we would stop here and say that $A$ is singular (there is no need to find rest cofactors).}

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad A_{22} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8, \quad A_{23} = -\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = 4,$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7, \quad A_{32} = -\begin{vmatrix} 0 & 2 \\ -2 & -1 \end{vmatrix} = -4, \quad A_{33} = \begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} = 2.$$

By formula (1)

$$A^{-1} = -\frac{1}{26} \begin{bmatrix} 3 & -1 & -7 \\ -2 & -8 & -4 \\ -12 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{26} & \frac{1}{13} & \frac{7}{26} \\ \frac{2}{13} & \frac{1}{13} & \frac{2}{13} \\ -\frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \end{bmatrix}.$$ 

The method of finding $A^{-1}$ using the augmented matrix $[A|I_3]$ seems to be faster for the previous example.

It worth mentioning that in case of $2 \times 2$ matrix $A$ formula (1) is especially simple:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) = ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
Make sure that $AA^{-1} = I_2$ (thus you will prove formula (1) for the case $n = 2$). For example,

$$
\begin{bmatrix}
2 & 1 \\
4 & 3
\end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix}
3 & -1 \\
-4 & 2
\end{bmatrix} = \begin{bmatrix}
\frac{3}{2} & -\frac{1}{2} \\
-2 & 1
\end{bmatrix}.
$$

Now describe the Cramer’s rule for solving linear systems $A\vec{x} = \vec{b}$. It is assumed that $A$ is a square matrix and $\det(A) \neq 0$ (or, what is the same, $A$ is invertible). Then, as we know, the linear system has a unique solution. The rule says that this solution is given by the formula

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

(2)

where $A_i$ is the matrix obtained from $A$ by replacing the $i^{th}$ column of $A$ by $\vec{b}$. [Don’t confuse with cofactors $A_{ij}$!]

**Example.** Solve the linear system

$$
\begin{align*}
3x_1 + x_2 - 2x_3 &= 4 \\
-x_1 + 2x_2 + 3x_3 &= 1 \\
2x_1 + x_2 + 4x_3 &= -2
\end{align*}
$$

We have (check all calculations!)

$$\det(A) = \begin{vmatrix}
3 & 1 & -2 \\
-1 & 2 & 3 \\
2 & 1 & 4
\end{vmatrix} = 35$$

Since $\det(A) \neq 0$, we can use the Cramer’s rule. Let’s find determinants of $A_1, A_2, A_3$:

$$
\begin{align*}
\det(A_1) &= \begin{vmatrix}
4 & 1 & -2 \\
1 & 2 & 3 \\
-2 & 1 & 4
\end{vmatrix} = 0, \\
\det(A_2) &= \begin{vmatrix}
3 & 4 & -2 \\
-1 & 1 & 3 \\
2 & -2 & 4
\end{vmatrix} = 70, \\
\det(A_3) &= \begin{vmatrix}
3 & 1 & 4 \\
-1 & 2 & 1 \\
2 & 1 & -2
\end{vmatrix} = -35.
\end{align*}
$$

Now by formula (2):

$$
\begin{align*}
x_1 &= \frac{0}{35} = 0, \\
x_2 &= \frac{70}{35} = 2, \\
x_3 &= \frac{-35}{35} = -1.
\end{align*}
$$

Thus $0, 2, -1$ is the solution to our system.

As before, in case of the linear system with two equations and two variables the solution is particularly simple. Consider the system

$$
\begin{align*}
ax + by &= e \\
(cx + dy) &= f
\end{align*}
$$

with unknowns $x$ and $y$. If $ad - bc \neq 0$, then by Cramer’s rule

$$
\begin{align*}
x &= \frac{de - bf}{ad - bc}, \\
y &= \frac{af - ce}{ad - bc}.
\end{align*}
$$

Make sure that these satisfy to the above system (thus you will prove Cramer’s rule for $2 \times 2$ case). For example, the system

$$
\begin{align*}
x + 3y &= 0 \\
2x + 7y &= 1
\end{align*}
$$

has the solution $x = -\frac{3}{1} = -3, \quad y = \frac{1}{1} = 1$. 

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