Lec 16: Cofactor expansion and other properties of determinants

We already know two methods for computing determinants. The first one is simply by definition. It works great for matrices of order 2 and 3. Another method is producing an upper-triangular or lower-triangular form of a matrix by a sequence of elementary row and column transformations. This can be performed without much difficulty for matrices of order 3 and 4. For matrices of order 4 and higher, perhaps, the most efficient way to calculate determinants is the cofactor expansion. This method is described as follows.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Denote by $M_{ij}$ the submatrix of $A$ obtained by deleting its row and column containing $a_{ij}$ (that is, row $i$ and column $j$). Then $\det(M_{ij})$ is called the minor of $a_{ij}$. For example, let

$$
A = \begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix}.
$$

(1)

$M_{11}$ is obtained by deleting row 1 and column 1; $M_{23}$ is $A$ without row 2 and column 3:

$$
M_{11} = \begin{vmatrix}
5 & 6 \\
8 & 9 \\
\end{vmatrix} \quad M_{23} = \begin{vmatrix}
1 & 2 \\
7 & 8 \\
\end{vmatrix}.
$$

The minor of $a_{11}$ is $\det(M_{11}) = 5\cdot 9 - 8\cdot 6 = -3$ and the minor of $a_{23}$ is $\det(M_{23}) = -6$.

If we multiply the minor of $a_{ij}$ by $(-1)^{i+j}$, then we arrive at the definition of the cofactor $A_{ij}$ of $a_{ij}$:

$$
A_{ij} = (-1)^{i+j} \det(M_{ij}).
$$

In the example above, $A_{11} = (-1)^2 \cdot (-3) = -3$, $A_{23} = (-1)^5 \cdot (-6) = 6$. Verify that $A_{12} = 6$, $A_{13} = -3$ and find the rest of cofactors.

The method of cofactor expansion is given by the formulas

$$
\det(A) = a_{1i}A_{1i} + a_{2i}A_{2i} + \cdots + a_{ni}A_{ni} \quad \text{(expansion of } \det(A) \text{ along } i^{th} \text{ row)}
$$

$$
\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \quad \text{(expansion of } \det(A) \text{ along } j^{th} \text{ column)}
$$

Let’s find $\det(A)$ for matrix (1) using expansion along the top row:

$$
\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1 \cdot (-3) + 2 \cdot 6 + 3 \cdot (-3) = 0.
$$

[Compare with the first example from the previous lecture. Basing on that example, could you say that $\det(A) = 0$ without any calculations?] It would be the same as if we used the expansion along any other row or column. For example, the expansion along the second column gives:

$$
\det(A) = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = -2 \begin{vmatrix}
4 & 6 \\
7 & 9 \\
\end{vmatrix} + 5 \begin{vmatrix}
1 & 3 \\
7 & 9 \\
\end{vmatrix} - 8 \begin{vmatrix}
1 & 3 \\
4 & 6 \\
\end{vmatrix} = 0.
$$

The method of cofactor expansion is especially applicable if a matrix has a row or a column with many zeros. Then we expand the determinant along this row or column.
Example. Compute the determinant of

\[
A = \begin{bmatrix}
2 & -1 & 1 & 0 \\
3 & 5 & 0 & -2 \\
1 & 1 & 0 & -3 \\
4 & 0 & 3 & -1
\end{bmatrix}.
\]

The third column looks more preferable as it contains two zeros. Let’s use the expansion along this column.

\[
\begin{vmatrix}
2 & -1 & 1 & 0 \\
3 & 5 & 0 & -2 \\
1 & 1 & 0 & -3 \\
4 & 0 & 3 & -1
\end{vmatrix} = 1 \cdot
\begin{vmatrix}
3 & 5 & -2 \\
1 & 1 & -3 \\
4 & 0 & -1
\end{vmatrix} - 3 \cdot
\begin{vmatrix}
2 & -1 \\
1 & 1 \\
3 & 5
\end{vmatrix} = -50 + 99 = 49.
\]

[We omitted zero terms.] Note that for computing the $3 \times 3$ determinants above we can use the expansion again. For example

\[
\begin{vmatrix}
3 & 5 & -2 \\
1 & 1 & -3 \\
4 & 0 & -1
\end{vmatrix} = 4 \begin{vmatrix} 5 & -2 \\ 1 & -3 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = -52 + 2 = -50.
\]

[We used the expansion along the bottom row.]

Important exercise: find $\text{det}(A)$ expanding along the second row and make sure the answer is the same.

Now let’s discuss some questions regarding determinants.

- What are the determinants of elementary matrices? They are $-1$, $r$ and $1$ for elementary matrices of respectively first, second (multiplication of a row or a column by $r$) and third type. For example, let $E$ be an elementary matrix corresponding to switching two rows. If we apply this ERT to the identity matrix $I_n$, we get $EI_n = E$. On the other hand, from the previous lecture we know that $\text{det}$ is multiplied by $-1$ after this transformation: $\text{det}(E) = -\text{det}(I_n) = -1$.

- Is it true that $\text{det}(rA) = r\text{det}(A)$? Yes — see the previous lecture.

- Is it true that $\text{det}(AB) = \text{det}(A)\text{det}(B)$? Yes. See the proof on pp. 151—153 of the book. As a consequence, the determinant of a product of any number of matrices is equal to the product of their determinants.

- Is it true that $\text{det}(A + B) = \text{det}(A) + \text{det}(B)$? No. If we take $A = I_2$, $B = -I_2$, then $\text{det}(A + B) = 0$ but $\text{det}(A) = \text{det}(B) = 1$, and $0 \neq 1 + 1 = 2$.

The property $\text{det}(AB) = \text{det}(A)\text{det}(B)$ is very important. It allows to prove

**Theorem.** Matrix $A$ is invertible if and only if $\text{det}(A) \neq 0$.

**Proof.** If $A$ is invertible, then it is a product of elementary matrices. Then, by the mentioned property, the determinant of $A$ is product of determinants of these matrices. Each of these determinants is nonzero as it must be $-1$, $r \neq 0$ or $1$. Therefore $\text{det}(A) \neq 0$. On the other hand, if $A$ is singular, its RREF $B = EA$ has a row of zeros, and $0 = \text{det}(B) = \text{det}(E)\text{det}(A)$. Since $\text{det}(E) \neq 0$, $\text{det}(A) = 0$. $\square$

If $A^{-1}$ exists, then $AA^{-1} = I_n$ and $\text{det}(A)\text{det}(A^{-1}) = \text{det}(I_n) = 1$. Hence $\text{det}(A^{-1}) = \frac{1}{\text{det}(A)}$. By theorem, matrix (1) is singular and the $4 \times 4$ matrix in the example above is invertible, with the determinant $\frac{1}{57}$.  
