MATH 2240 MIDTERM, SPRING 2015

Name, written slowly and legibly: ________________________________

In each answer, write as much (on front and back) as it takes to convey your thought process; full English sentences are much easier to give credit to than bare, unmotivated scribbled formulæ. (They won’t do any good if they can’t be read, so do put effort into making them legible.)

Feel free to ask me questions during the test, especially if you need a little reminder about a definition. Worst case is I don’t answer. (It’s very sad to afterward hear “I didn’t realize I could ask you that” — find out!)

1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded function, and \( I_1 = [a_1, b_1] \), \( I_2 = [a_2, b_2] \), \ldots a sequence of intervals.

a [15 pts]. If \( f \) is Riemann integrable, show that for any \( \epsilon > 0 \), there exists such a sequence of intervals such that

\[
\sum_k 1_{I_k} \inf_{I_k} (f) \leq f \leq \sum_k 1_{I_k} \sup_{I_k} (f)
\]

pointwise

and

\[
\sum_k (b_k - a_k)(\sup_{I_k} (f) - \inf_{I_k} (f)) < \epsilon
\]

where \( \inf, \sup \) are the greatest lower and least upper bounds of \( f \) on \( I_k \).

Answer. If \( f \) is Riemann integrable, it’s of bounded support inside some interval \([-2^n, 2^n)\), and we can take for \((I_k)\) the finite sequence of \( N \)-dyadic intervals inside there.

By the definition of Riemann integrability, as \( N \to \infty \) the difference in this second line goes to 0.

1b [10 pts]. Show the converse of (a) fails, i.e., find a bounded function \( f \) such that for any \( \epsilon \) we have such a sequence of intervals, but \( f \) isn’t Riemann integrable.

Answer. Let \( f(x) = 58 \) for all \( x \), and \( I_k = [n_k, n_k + 1) \) where \((n_k)\) is an enumeration of the integers. Then the inequalities are strict and the second sum is 0. But of course \( f \) isn’t integrable.

2. Let \( f \) be Riemann integrable. Define \( g \) by

\[
g(x) = \sum_{n \in \mathbb{N}} \frac{f(2^n x)}{2^n}
\]

a [10 pts]. Show \( g \) is Lebesgue integrable.

Answer. By the change of variable formula, \( \int |f(2^n x)| = \frac{1}{2^n} \int |f(x)| \), so \( \sum_n \int |f(2^n x)| = (\sum_n \frac{1}{2^n}) \int |f(x)| < \infty \). Then use our definition of Lebesgue-integrable function.
b [20 pts]. Show \( g \) is Riemann integrable. (Hint: don’t use the definition; we have a killer theorem about when this is true.)

**Answer.** If \( f \) is supported within a ball of radius \( r \), so is \( g \).

If \( |f| \leq c \) everywhere, then \( |g(x)| \leq \sum_n \frac{c}{2^n} \leq 2c \).

If \( S(f) \) is the set of points where \( f \) is discontinuous, then we claim that

\[
S(g) \subseteq (S(f) \cup S(f)/2 \cup S(f)/4 \cup \ldots) \cup \{0\}
\]

Why? Outside of any ball of radius \( \epsilon \), \( g = \sum_{n \leq \log_2(\tau/\epsilon)} \frac{f(2^n)}{2^n} \) so can only be discontinuous where those functions are. Since each of those sets \( S(f)/2^k \) is measure zero and there are only countably many of them, \( S(g) \) is measure zero.

Hence \( g \) is bounded with bounded support and continuous almost everywhere, so Riemann integrable.

3 [15 pts]. Find a **continuous** function \( f(x,y) \) on \( \mathbb{R}^2 \) such that the iterated Riemann integrals

\[
\int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(x,y)
\]

exist but the iterated Riemann integrals

\[
\int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} f(x,y)
\]

don’t, i.e. *something* goes wrong in the latter. Did I mention \( f \) should be **continuous**?

**Answer.**

\[
f(x,y) = \begin{cases} 
sin(y) & \text{if } y \in [0, 2\pi] \\
0 & \text{otherwise}
\end{cases}
\]

Then each \( \int_{y \in \mathbb{R}} f(x,y) = 0 \), and those 0s can be integrated over \( x \), too. But if we attempt \( \int_{x \in \text{reals}} f(x,y) \) for \( y = \pi/2 \) say, we’re looking at \( \int_{x \in \text{reals}} 1 \) which isn’t integrable.

4. Define \( \Phi : (0,1) \to (1,\infty) \) by \( \Phi(x) = \frac{1}{x} \).

a [10 pts]. Given a function \( f : (1,\infty) \to \mathbb{R} \), write down the change-of-variable formula for integrating \( f \) vs. \( f \circ \Phi \).

**Answer.** \( D\Phi|_x = \left[ -\frac{1}{x^2} \right] \), so \( |\det D\Phi|_x = \frac{1}{x^2} \). Hence

\[
\int_{x \in (0,1)} \frac{f(\Phi(x))}{x^2} = \int_{y \in (1,\infty)} f(y).
\]

b [10 pts]. Find a function \( f \) such that exactly one of the integrals from part (a) exists as a Riemann integral. (The other side will only be Lebesgue integrable.)

**Answer.** Let’s make the left integral be of 1, i.e. \( f(\Phi(x))/x^2 = 1 \). Then \( x^2 = f(\Phi(x)) = f(1/x) \), or \( 1/y^2 = f(y) \). Now the RHS is \( \int_{y \in (1,\infty)} \frac{1}{y^2} \), which isn’t of bounded support so isn’t a Riemann integrable function. But the LHS is fine.
c [10 pts]. In part (b), why can’t you make the other one of the two integrals be the only one that exists as a Riemann integral?

*Answer.* Say the RHS exists as a Riemann integral. Then \( f \) is of bounded support, and in particular \( f(y) = 0 \) for \( y > b \) (for some \( b \)). Hence \( (f \circ \Phi)(x) = 0 \) for \( x < 1/b \).

Since \( |f| \leq c \) for some \( c \) on \((1, \infty)\), we learn that on \((0, 1)\) that \( |f \circ \Phi| \leq c \), and \( |(f \circ \Phi)/x^2| \leq b^2c \).

Since \( f \) is continuous a.e., so is \( f \circ \Phi \). So the LHS is Riemann integrable too.